Event-Triggered Intermittent Sampling for Nonlinear Model Predictive Control

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Abstract

In this paper, we propose a new aperiodic formulation of model predictive control for nonlinear continuous-time systems. Unlike earlier approaches, we provide event-triggered conditions without using the optimal cost as a Lyapunov function candidate. Instead, we evaluate the time interval when the optimal state trajectory enters a local set around the origin. The obtained event-triggered strategy is more suitable for practical applications than the earlier approaches in two directions. Firstly, it does not include parameters (e.g., Lipschitz constant parameters of stage and terminal costs) which may be potential source of conservativeness or may not be explicitly known depending on the problem formulation. Secondly, the event-triggered conditions are necessary to be checked only at sampling time intervals, instead of continuously. This leads to the alleviation of both sensing cost and computational burden to evaluate event-triggered conditions. The proposed event-triggered scheme is also validated through numerical simulations.

Key words: Event Triggered Control, Self-Triggered Control, Nonlinear Model Predictive Control

1 Introduction

Event-Triggered Control (ETC) and Self-Triggered Control (STC) have been active areas of research in the community of networked control systems, due to their potential advantages over the typical time-triggered controllers \cite{1,2,3,4,5,6}. In contrast to the time-triggered case where the control signals are executed periodically, ETC and STC trigger the executions based on the violation of certain prescribed control performances, such as Input-to-State Stability (ISS) \cite{3}, $L_2$ gain stability \cite{2}, and LMI based stability conditions \cite{5}. The major advantage of these methods is to alleviate over-usage of communication resources and the energy consumption, by transmitting control signals only when it is necessary.

In another line of research, Model Predictive Control (MPC) has been one of the most popular control strategies applied in a wide variety of applications, such as process industry \cite{7}, networked control systems \cite{8}, multiagent systems \cite{9}, and robotics applications \cite{10}. MPC plays an important role for the real system when several constraints, such as actuator or physical limitations, need to be explicitly taken into account. The basic idea of MPC is to obtain the current control action by solving the Optimal Control Problem (OCP) online, based on the knowledge of current state measurement and future behavior prediction through the dynamics.

In traditional MPC, only the current portion of the optimal control input is applied to the plant, and the OCP is periodically solved with a given specific time interval. However, this may not be preferable from a computational view point, since solving numerical OCPs at every time instant may require a high computational load. Motivated this, it is useful to employ the event-triggering mechanisms; the OCP is solved only when some events, generated based on certain control performance criteria, are triggered. The application of ETC or STC to MPC, generally known as Event-Triggered MPC (ETMPC) and Self-triggered MPC (STMPC), has been pursued in recent years; most of the works focus on discrete-time systems, see \cite{11,12,13,14} for the linear and \cite{15,16,17} for the nonlinear case, and some results include \cite{18,19,20,21} for the continuous case. In this paper, we are particularly interested in the case of continuous-time nonlinear systems with additive disturbances. In \cite{18}, a self-triggered strategy was proposed for nonholo-
nomic systems. The self-triggered condition was derived based on the optimal cost regarded as an ISS Lyapunov function candidate. In [19], an event-triggered strategy has been proposed for general nonlinear systems with additive bounded disturbances. When deriving the event-triggered strategy, an additional state constraint is imposed such that the optimal cost as a Lyapunov function candidate is decreasing. In [21], a self-triggered strategy was provided for nonlinear input affine systems. Although the self-triggered rules were derived similarly to the previous works of STMPC, an additional way to discretize an optimal control trajectory into several control samples was provided so that these can be transmitted to the plant over the network channels.

In this paper, we propose a new event-triggered formulation of MPC for nonlinear continuous-time systems. In contrast to the earlier results illustrated above, the event-triggered strategy is derived based on a new stability theorem, which does not evaluate the optimal cost as a Lyapunov function candidate. In the stability derivations, we instead evaluate the time interval when the optimal state trajectory enters a local region around the origin. By guaranteeing that this time interval becomes smaller as the OCP is solved, it is ensured that the state enters a prescribed set in network channels.

The derivation of the new stability is motivated by the fact that the conventional event-triggered strategies (derived from the optimal cost) may include Lipschitz constant parameters for the stage and terminal cost, see e.g., [18,21]. Since these parameters are characterized by the maximum distance of the state from the origin, the triggering condition becomes largely affected by the state domain considered in the problem formulation. That is, as a larger state domain is considered, the event-triggered condition becomes more conservative. Furthermore, if the exact state domain is not known (e.g., if there exists no physical limitations), these parameters are not known explicitly. Depending on the problem formulation, therefore, it is not desirable to include these parameters in the event-triggered condition.

Since our approach does not evaluate the optimal cost as a Lyapunov function candidate, the corresponding event-triggered conditions do not include such unsuitable parameters. We will also illustrate through a simulation example that the proposed approach attains much less conservative result than our previous result presented in [21].

The idea of not using the optimal cost follows our preliminary approach presented in [20]. However, the proposed method differs from the previous work in the following two directions. Firstly, the stability is proven in a different manner from the one analyzed in [20]. Aside from the event-triggered strategy, an important result of this paper is that we can explicitly obtain the maximum time of convergence: in contrast to the existing stability of MPC which only illustrates that, “the state converges to a local set in finite time”, our approach additionally derives, “how long it will (maximally) take to achieve convergence”. Although the analysis of convergence time has been also conducted for linear discrete-time systems, e.g., [22], this paper derives it for nonlinear continuous-time systems with additive bounded disturbances. Regarding the control performance, we also point out that there exists a trade-off between the time of convergence and robustness to the size of disturbances.

Secondly, as another main contribution of this paper, we will propose a new event-triggered framework, which does not require continuous state measurements and evaluations of the event-triggered conditions. In the proposed strategy, the triggering conditions are evaluated only at certain time intervals. This approach is clearly advantageous over the existing ET MPC strategies, since it also alleviates a sensing cost and a computational load to evaluate the event-triggered conditions. Moreover, we also illustrate that there exists a trade-off between the number of state measurements and the number of OCPs, and this can be regulated through an appropriate parameter tuning.

The proposed framework is relevant to Periodic Event-Triggered Control (PETC), which has been recently investigated for linear systems with state-feedback controllers, see e.g., [4,5]. In the general PETC, the sampling time interval to evaluate the event-triggered condition is constant for all update times. In our proposed approach, on the other hand, the sampling time intervals are selected in an adaptive way: at each update time of solving OCP, the controller adaptively determines the sampling time interval to check the event-triggered condition, such that the desired control performance can be guaranteed.

As an illustrative example, we consider a problem of controlling a spring-mass system, and show that the state converges to a local set around the origin within a prescribed time interval according to the stability analysis.

This paper is organized as follows. In Section 2, the optimal control problem is formulated. In Section 3, feasibility of the OCP is analyzed. In Section 4, our main proposed algorithm is presented, and the stability is shown in Section 5. A simulation example validates our proposed method in Section 6. We finally conclude in Section 7.

Notations. We let \( \mathbb{R} \), \( \mathbb{R}_{\geq 0} \), \( \mathbb{N}_{\geq 0} \), \( \mathbb{N}_{\geq 1} \) be the real, non-negative real, non-negative integers and positive integers, respectively. For a matrix \( Q \), we use \( Q > 0 \) to denote that \( Q \) is positive definite. We denote \( \| x \| \) as the Euclidean norm of vector \( x \), and \( \| x \|_p \) as a weighted norm of vector \( x \), i.e., \( \| x \|_p = \sqrt{x^T P x} \). Given a compact set \( \Phi \subseteq \mathbb{R}^n \), we denote \( \partial \Phi \) as the boundary of \( \Phi \). The function \( f(x,u) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is called Lips-
chitz continuous with a weighted matrix $P$, if there exists $0 \leq L_f < \infty$ such that for all $x_1, x_2 \in \Omega$, it holds that $\|f(x_1, u) - f(x_2, u)\|_p \leq L_f \|x_1 - x_2\|_p$.

2 Problem formulation

2.1 Dynamics and optimal control problem

In this section the problem formulation is defined. We aim at applying MPC to the following nonlinear systems with additive disturbances:

$$\dot{x}(t) = f(x(t), u(t)) + w(t)$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $w(t) \in \mathbb{R}^n$ is an additive bounded disturbance. The control input $u$ and the disturbance $w$ are assumed to satisfy the following constraints:

$$u(t) \in U \subseteq \mathbb{R}^m, \quad w(t) \in W \subseteq \mathbb{R}^n.$$  

(2)

where $U$ and $W$ are compact subsets containing the origin in their interiors. For the plant model given by (1), it is assumed that the followings are satisfied:

**Assumption 1** The nonlinear function $f(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is twice continuously differentiable, and $f(0, 0) = 0$. Furthermore, for the linearized system around the origin with no disturbances:

$$\dot{x}(t) = A_f x + B_f u,$$

(3)

where $A_f = \partial f/\partial x(0,0)$ and $B_f = \partial f/\partial u(0,0)$, the pair $(A_f, B_f)$ is stabilizable.

The assumption of the stabilizability of the linearized system is utilized to guarantee the existence of a positive invariant set around the origin. Let $t_k$, $k \in \mathbb{N}_{\geq 0}$ be the update instants when OCPs are solved, and let $\Delta_k = t_{k+1} - t_k$ be the inter-event times. At $t_k$, the controller solves an OCP based on the state measurement $x(t_k)$ and the predictive behavior of the systems described by (1). In this paper, we consider the following cost to be minimized:

$$J(x(t_k), u(\cdot)) = \int_{t_k}^{t_k+T_k} \|\dot{\hat{x}}(\xi)\|^2_Q + \|w(\xi)\|^2_P d\xi,$$

(4)

where $Q = Q^T \succ 0$, $R = R^T \succ 0$ and $T_k > 0$ is the prediction horizon. $\dot{\hat{x}}(\xi)$ denotes the nominal trajectory of (1) given by

$$\dot{\hat{x}}(\xi) = f(\hat{x}(\xi), u(\xi))$$

(5)

for all $\xi \in [t_k, t_k + T_k]$ with $\hat{x}(t_k) = x(t_k)$. Although the prediction horizon is given constant for any update times in the standard formulation of MPC, we consider here that $T_k$ is adaptively selected based on the previous results of OCPs. This variable horizon strategy will be a key idea to prove stability without using the optimal cost as a Lyapunov function candidate. More characterization of $T_k$ is provided in this section when formulating the OCP.

Similarly to the MPC strategy presented in [23] it is shown that the following lemma holds for the existence of a local controller:

**Lemma 1** Suppose that Assumption 1 holds. Then, there exists a positive constant $0 < \varepsilon < \infty$, a matrix $P_f = P_f^T > 0$, and a local controller $\kappa(x) = Kx \in U$, satisfying

$$\frac{\partial V_f}{\partial x} f(x, \kappa(x)) \leq -\frac{1}{2} v^T (Q + K^T R K) v$$

(6)

for all $x \in \Phi$, where $V_f(x) = x^T P_f x$ and $\Phi = \{x \in \mathbb{R}^n : V_f(x) \leq \varepsilon^2\}$. Furthermore, $\Phi$ is a positive invariant set for the system (1) with $\kappa(x) = K x \in U$, if the disturbance $w$ satisfies $\|w\|_{P_f} \leq \hat{w}_{\text{max}}$, where

$$\hat{w}_{\text{max}} = \frac{\varepsilon}{4 \lambda_{\text{max}}(Q_P)}.$$  

(7)

The proof is obtained by extending Lemma 1 in [23] and is given in the Appendix.

**Definition 1** (Control Objective of MPC) The control objective of MPC is to steer the state $x$ to the local region $\Phi$ in finite time.

In this paper, we consider that the control law switches from applying MPC to the utilization of the local controller $\kappa$, as soon as the state enters $\Phi$. This switching control law is referred to as ‘Dual mode MPC’, which is adopted in many works in the literature (see, e.g., [24]). Based on the local set $\Phi$, we further define the restricted set $\Phi_f$ given by

$$\Phi_f = \{x \in \mathbb{R}^n : V_f(x) \leq \varepsilon_f^2\},$$

where $0 < \varepsilon_f < \varepsilon$. Since $\varepsilon_f < \varepsilon$, the set $\Phi_f$ is contained in $\Phi$, i.e., $\Phi_f \subseteq \Phi$. An example of these regions is illustrated in Fig. 1.

**Assumption 2** The nonlinear function $f(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous with the weighted matrix $P_f$, with the Lipschitz constant $0 \leq \varepsilon_f < \infty$.

Assumption 2 will be used to derive several conditions to guarantee feasibility of the OCP. In the formulation
posed event-triggered strategy. The closed-loop system of MPC, we iteratively find at each update time \( t_k, k \in \mathbb{N}_{\geq 0} \), an optimal state \( \hat{x}^*(\xi) \) and a control trajectory \( u^*(\xi) \) for all \( \xi \in [t_k, t_k + T_k] \), by minimizing the cost given by (4). Regarding constraints, we impose that the optimal state reaches \( \Phi_f \) within the prediction horizon \( T_k \), i.e., \( \hat{x}^*(t_k + T_k) \in \Phi_f \). Since \( \hat{x}^*(t_k + T_k) \in \Phi_f \), there exists a positive time interval when the optimal state enters the boundary of \( \Phi_f \). We let \( T_k^* (T_k^* \leq T_k) \) be such time interval obtained at \( t_k \), i.e., \( \hat{x}^*(t_k + T_k^*) \in \partial \Phi_f \). The time interval \( T_k^* \) is illustrated also in Fig. 1.

Based on the above notations, we propose the following OCP:

**Problem 1 (OCP)** For the non-initial time \( t_k, k \in \mathbb{N}_{\geq 1} \), given \( x(t_k) \) and \( T_{k-1}^* \), the OCP is to minimize the cost \( J(x(t_k), u(\cdot)) \) given by (4), subject to

\[
\begin{align*}
\dot{x}(\xi) &= f(\tilde{x}(\xi), u(\cdot)), \quad \xi \in [t_k, t_k + T_k] \\
0 &= \hat{x}(t_k + T_k) \in \Phi_f,
\end{align*}
\]

(8) \( u(\cdot) \in U \) \( x(t_0 + T_0) \in \Phi_f \) for a given \( t_0 > 0 \).

In contrast to the standard terminal constraint with a constant prediction horizon, we require by (10) that the optimal state trajectory enters \( \Phi_f \) within \( T_k = T_{k-1}^* - \alpha \Delta_k \), where \( T_{k-1}^* \) is the time interval obtained by the previous OCP at \( t_{k-1} \). This means that the obtained \( T_k^* \) satisfies \( T_k^* \leq T_k = T_{k-1}^* - \alpha \Delta_k < T_{k-1}^* \), which guarantees that the time to reach \( \Phi_f \) becomes strictly smaller than the previous one at \( t_{k-1} \). In later sections, we will make use of this property to show that the state trajectory enters \( \Phi \) in finite time.

When applying MPC, we consider that the optimal control input trajectory \( u^*(\xi) \) is applied until the next update time \( t_{k+1} \), where \( t_{k+1} \) is determined by the proposed event-triggered strategy. The closed-loop system for \( t \in [t_k, t_{k+1}) \) is thus given by

\[
\dot{x}(t) = f(x(t), u^*(t)) + w(t), \quad t \in [t_k, t_{k+1}).
\]

### 3 Feasibility analysis

One of the most important properties of MPC is to ensure the existence of a solution to the OCP defined in Problem 1. The main result of this section is thus to provide several conditions to guarantee recursive feasibility, which states that the existence of a feasible solution at an initial update time \( t_0 \) implies the feasibility at any update times afterwards \( t_k, k \in \mathbb{N}_{\geq 1} \). The obtained feasibility conditions will be key ingredients to derive the event-triggered strategy, which will be discussed in the next section.

**Theorem 1** Suppose that the OCP defined in Problem 1 has a solution at \( t_k \), providing an optimal control input \( u^*(\xi) \) and the corresponding state trajectory \( \hat{x}^*(\xi) \) for all \( \xi \in [t_k, t_k + T_k] \), and the time interval \( T_k^* \). Then, Problem 1 has a solution at \( t_{k+1} (> t_k) \), if the followings are satisfied:

\[
\begin{align*}
\Delta_k &= t_{k+1} - t_k \leq T_k^*, \\
||x(t_{k+1}) - \hat{x}^*(t_{k+1})||_P &\leq (\varepsilon - \varepsilon_f) e^{-L_f T_k^*}, \\
||w||_{P_f} &\leq \bar{w}_{\text{max}},
\end{align*}
\]

where \( \bar{w}_{\text{max}} \) is given by

\[
\bar{w}_{\text{max}} = \frac{\lambda \min (Q_f)}{4 e^{L_f T_k^*}} (1 - \alpha \varepsilon_f)
\]

**PROOF.** Consider the following feasible control trajectory candidate:

\[
\bar{u}(\xi) = \begin{cases} 
  u^*(\xi), & \xi \in [t_k, t_k + T_k] \\
  \kappa(\tilde{x}(\xi)), & \xi \in (t_k + T_k^*, t_{k+1} + T_k^*] 
\end{cases}
\]

(16) where \( T_{k+1} = T_k^* - \alpha \Delta_k \). Here we have \( t_{k+1} + T_{k+1} > t_k + T_k^* \) since

\[
t_{k+1} + T_{k+1} = t_k + \Delta_k + T_k^* - \alpha \Delta_k \\
= t_k + \Delta_k + T_k^* - \alpha \Delta_k > t_k + T_k^*
\]

\( \tilde{x}(\xi) \) denotes the predictive state trajectory obtained by applying \( \bar{u}(\xi) \), i.e., \( \tilde{x}(\xi) = f(\tilde{x}(\xi), \bar{u}(\xi)) \) with \( \tilde{x}(t_{k+1}) = x(t_k) \).

To prove that (16) is a feasible controller, we show that the following three arguments are satisfied:

(i) By applying \( \bar{u}(\xi) \), \( \xi \in [t_k, t_k + T_k^*] \), the predictive state enters \( \Phi \) by the time \( t_k + T_k^* \). That is,
We now prove the statement given in (iii). By using $\Delta_k$ when $t + T_k^*$ is admissible.

By applying $\dot{u}(\xi)$, $\xi \in [t_k + T_k^*, t_{k+1} + T_{k+1}]$, the predictive state $\tilde{x}$ enters $\Phi_f$ by the time $t_{k+1} + T_{k+1}$. That is, 
$$\tilde{x}(t_{k+1} + T_{k+1}) \in \Phi_f$$

To prove (i), we first use the fact that the difference between $\tilde{x}$ and $\tilde{x}^*$ is upper bounded by
$$||\tilde{x}(\xi) - \tilde{x}^*(\xi)||_{\mathcal{F}} \leq ||\tilde{x}(t_{k+1}) - \tilde{x}^*(t_{k+1})||_{\mathcal{F}} e^{L_f(\xi - t_{k+1})}$$
for $\xi \in [t_{k+1}, t_k + T_k^*]$. Supposing that (13) holds and letting $\xi = t_k + T_k^*$, we obtain
$$||\tilde{x}(t_k + T_k^*) - \tilde{x}^*(t_k + T_k^*)||_{\mathcal{F}} \leq e^{-L_f T_k}(\varepsilon - \varepsilon_f) e^{-L_f(t_k + T_k^* - t_k)}$$
$$= (\varepsilon - \varepsilon_f) e^{-L_f(t_k + T_k^*)}$$

From the triangular inequality, we obtain
$$||\tilde{x}(t_k + T_k^*)||_{\mathcal{F}} \leq ||\tilde{x}^*(t_k + T_k^*)||_{\mathcal{F}} + (\varepsilon - \varepsilon_f) e^{-L_f(t_k + T_k^*)}$$
$$\leq \varepsilon_f + \varepsilon - \varepsilon_f = \varepsilon$$

It holds that $\tilde{x}(t_k + T_k^*) \in \Phi$ and the proof of (i) is completed.

The proof of (ii) is obtained from the fact that we have $\Delta_k \leq T_k^*$ from the event-triggered strategy, and thus $T_k^* - \alpha \Delta_k \geq (1 - \alpha)T_k^* > 0$.

We now prove the statement given in (iii). By using $\tilde{x}(t_k + T_k^*) \in \Phi$ and from (6), we obtain
$$\dot{V}_f(\tilde{x}(\xi)) \leq -\frac{1}{2} \tilde{x}^T(\xi)(Q + K^T R K)\tilde{x}(\xi)$$
$$\leq -\frac{1}{2} \lambda_{\min}(\hat{Q}_P) V_f(\tilde{x}(\xi))$$
for $\xi \in [t_k + T_k^*, t_{k+1} + T_k^* - \alpha \Delta_k]$. Furthermore, from the Gronwall-Bellman inequality and by supposing that (14) holds, we obtain
$$||\tilde{x}(t_k + T_k^*)||_{\mathcal{F}} \leq \frac{\tilde{x}^T(\xi)(Q + K^T R K)\tilde{x}(\xi)}{\lambda_{\min}(\hat{Q}_P)(1 - e^{-L_f \Delta_k})}$$

Denoting $\eta = \frac{(1 - \alpha)}{4L_f} \lambda_{\min}(\hat{Q}_P)$, and by using comparison lemma, we obtain
$$V_f(\tilde{x}(t_{k+1} + T_k^* - \alpha \Delta_k))$$
$$\leq V_f(\tilde{x}(t_k + T_k^*)) e^{-0.5 \lambda_{\min}(\hat{Q}_P)(1 - \alpha)\Delta_k}$$
$$\leq \varepsilon_f^2 (1 + \eta(1 - e^{-L_f \Delta_k}))^2 e^{-2L_f \alpha \Delta_k}$$
$$\leq \varepsilon_f^2$$

The 3rd inequality is obtained by the fact that the function $g_k(\Delta_k) = (1 + \eta(1 - e^{-L_f \Delta_k})) e^{-L_f \alpha \Delta_k}$ is shown to be a decreasing function of $\Delta_k$ with $g_k(0) = 1$. Thus we obtain $V_f(\tilde{x}(t_{k+1} + T_k^* - \alpha \Delta_k)) \leq \varepsilon_f^2$, and the proof of (iii) is completed.

Based on above, the controller given by (16) provides a feasible solution to Problem 1 for $t_{k+1} > t_k$, provided that the conditions (12), (13), and (14) are satisfied. This completes the proof.

4 Event-triggered intermittent sampling strategy

By making use of the feasibility result provided in the previous section, we now propose an event-triggered strategy. Suppose again that the OCP is solved at $t_k$, providing a pair of optimal control $u^*(\xi)$ and the corresponding state $\tilde{x}^*(\xi)$ for all $\xi \in [t_k, t_k + T_k]$. Through an event-triggered condition, we consider to determine the next OCP update time $t_{k+1} > t_k$ such that the feasibility is ensured.

The simplest way to determine $t_{k+1}$ might be to use the original feasibility conditions directly as the event-triggered conditions. That is, for each $t > t_k$, check the feasibility according to (12) and (13), i.e.,
$$||\tilde{x}(\xi) - \tilde{x}^*(\xi)||_{\mathcal{F}} \leq (\varepsilon - \varepsilon_f) e^{-L_f T_k^*},$$
$$t - t_k \leq T_k^*.$$ (19)

Only when either of the above conditions is violated, then we set $t_{k+1} = t$ as the next update time. This strategy ensures the feasibility of the OCP and may reduce a computational load of solving OCPs. However, checking the above conditions for each $t > t_k$ requires continuous monitoring of the state $x(t)$ and evaluations of the above conditions, which clearly leads to a high cost of sensing and a computation load.

Therefore, we propose here an alternative approach by relaxing the above continuous requirements. The key idea of our approach is to measure the state and evaluate event-triggered conditions only at certain sampling time intervals, instead of continuously. A schematic block diagram of our proposed scheme is illustrated in Fig. 2.
If we would directly use (18) as the event-triggered condition is evaluated. Namely, from the obtained $\delta_k^*$ from SDS, the Event-Triggered System (ETS) measures the state and checks the event-triggered condition only at $t_k + m\delta_k^*$, $m \in \mathbb{N}_{\geq 1}$, in order to determine the next update time $t_{k+1}$. Note that the SDS has a partial role to determine $t_{k+1}$ to solve the OCP (the black dotted arrow in Fig. 2); as described later in this section, $t_{k+1}$ can sometimes be directly determined according to $T_k^*$ without needing to evaluate the event-triggered condition.

Regarding the proposed framework outlined above, we need to derive both mechanisms to determine $\delta_k^*$ and the event-triggered conditions. One might directly utilize (18), (19) as the event-triggered conditions, and evaluate them with a given arbitrary value of $\delta_k^*$. However, this cannot be applied due to the following two problems regarding the violation of feasibility:

(P.1) If a large value of $\delta_k^*$ would be chosen, the feasibility would not be satisfied at the next evaluation time $t_k + \delta_k^*$. We illustrate this issue in Fig. 3.

(P.2) If we would directly use (18) as the event-triggered condition, the feasibility might be violated between two consecutive evaluation times. This issue is illustrated in Fig. 4. The critical problem here is that the controller does not know whether the feasibility is violated between two evaluation times; when arriving at a certain evaluation time (e.g., green cross mark in Fig. 4), it is possible that the error $\|x(t) - \hat{x}^*(t)\|$ already exceeds the threshold, and a loss of feasibility occurs.

In the following, we provide solutions to each problem above and then provide the over-all event-triggered strategy. Consider first to solve (P.1). In order to deal with the problem, $\delta_k^*$ needs to be chosen small enough such that the feasibility is guaranteed for all $t \in [t_k, t_k + \delta_k^*]$. Thus we evaluate a minimum inter-event time of the feasibility conditions given by (18),

\[
\|x(t) - \hat{x}^*(t)\|_{P_f} \leq \frac{\hat{w}_{\text{max}}}{L_f} (e^{L_f(t-t_k)} - 1) - 1
\]  

for $t \in [t_k, t_k + T_k]$. Thus, a sufficient condition to satisfy (18) is

\[
\lambda_{\text{min}}(\tilde{Q}_P)(1 - \alpha)e_f e^{L_f(t-t_k)} - 1 \leq (\varepsilon - \varepsilon_f)e^{-L_fT_k^*}
\]

Solving the above for $t$ yields $t \leq t_k + \Delta_k^{\text{min}}$, where $\Delta_k^{\text{min}}$
\[ \Delta_k^{\text{min}} = \frac{1}{L_f} \ln \left( 1 + \frac{2L_f(\varepsilon - \varepsilon_f)e^{L_f(T_k^* - T_k)} - \varepsilon_f}{\lambda_{\min}(Q_P)(1 - \alpha)\varepsilon_f} \right) > 0 \]  
\text{(21)}

This implies that the condition (18) is satisfied for all \( t \in [t_k, t_k + \Delta_k^{\text{min}}] \). By taking into account the other feasibility condition (19), the over-all minimum inter-event time is now given by \( \min\{\Delta_k^{\text{min}}, T_k^*\} \). For the case we have \( \Delta_k^{\text{min}} \leq T_k^* \), the minimum inter-event time becomes \( \Delta_k^{\text{min}} \). Thus, if the sampling time interval \( \delta^*_k \) is selected such that \( \delta^*_k = \gamma \Delta_k^{\text{min}} \leq \Delta_k^{\text{min}} \) for a given \( 0 < \gamma \leq 1 \), the feasibility is guaranteed for all \( t \in [t_k, t_k + \delta^*_k] \). On the other hand, if the feasibility at \( t^* = T_k^* \) is satisfied for all \( t \in [t_k, t_k + T_k^*] \). This means that (19) is violated earlier than (18). Thus, for the case we have \( T_k^* < \Delta_k^{\text{min}} \), we can directly set the next time as \( t_{k+1} = t_k + \Delta_k^{\text{min}} \).

Based on the above analysis, the following strategy can be provided as a solution to (P.1), this can serve as the role of the SDS.

(i) If \( T_k^* \geq \Delta_k^{\text{min}} \), then set \( \delta^*_k = \gamma \Delta_k^{\text{min}} \) for a given \( 0 < \gamma \leq 1 \).

(ii) If \( T_k^* < \Delta_k^{\text{min}} \), then set \( t_{k+1} = t_k + \Delta_k^{\text{min}} \) as the next update time.

Next, we solve (P.2) by modifying the original feasibility conditions. Based on the obtained \( \delta^*_k \), the time instants to measure the state and evaluate the event-triggered condition are now given by \( t_k + m\delta^*_k, m \in \mathbb{N}_{\geq 1} \). To avoid losing the feasibility between two evaluation times, we check the feasibility condition at one step future time, instead of the current time instant. That is, at an evaluation time \( t = t_k + m\delta^*_k, m \in \mathbb{N}_{\geq 1} \), the state \( x(t) \) is measured and then the feasibility is checked for \( t + \delta^*_k \) instead of \( t \). If the feasibility at \( t + \delta^*_k \) is guaranteed, then we move on to the next evaluation time \( t + \delta^*_k \). On the other hand, if the feasibility at \( t + \delta^*_k \) is not guaranteed, then we set \( t_{k+1} = t \). Since we preliminary check the feasibility at one step future time, the loss of feasibility does not occur between two evaluation times. The illustration of this scheme is depicted in Fig. 5.

The feasibility at one step future time can be given by modifying the original feasibility conditions. Suppose at an evaluation time \( t = t_k + m\delta^*_k, m \in \mathbb{N}_{\geq 1} \), we aim at checking the feasibility at \( t + \delta^*_k \) based on the state measurement \( x(t) \). The difference between the actual state and the optimal state at \( t + \delta^*_k \) is given by

\[ \|x(t) - \hat{x}(t)\|_P \leq \varepsilon_f e^{-L_f(T_k^* + \delta^*_k)} - \lambda_{\min}(Q_P)(1 - \alpha)\varepsilon_f(1 - e^{-L_f\delta^*_k}) \]

\text{(24)}

Note that if (24), (25) are both satisfied the feasibility is guaranteed at \( t + \delta^*_k \), and these conditions can be evaluated based on \( x(t) \). Therefore, by using (24) and (25) as the event-triggered conditions, the violation of the feasibility between two evaluation times will not occur, providing thus a solution to (P.2).

Based on the above results, the over-all proposed algorithm of the event-triggered strategy is summarized below:
Algorithm 1 (Event-triggered intermittent sampling for MPC):

(i) At any update times $t_k, k \in \mathbb{N}_{\geq 0}$, if $x(t_k) \in \Phi$, then switch to the local controller $\kappa(x)$ as a dual mode strategy. Otherwise, solve Problem 1 and obtain the optimal control trajectory $u^*(\xi)$ and the corresponding state $\hat{x}^*(\xi)$ for all $\xi \in [t_k, t_k + T_k]$. Then, calculate $t_k^+$ as the time interval when the state reaches $\Phi_f$, i.e., $\hat{x}^*(t_k + T_k^+) \in \partial \Phi_f$.

(ii) The ETS provides the sampling time $\delta_k^*$ or the next update time $t_{k+1}$ in the following way:

(a) If $T_k^+ \geq \Delta_k^{\min}$, where $\Delta_k^{\min}$ is given by (21), then set $\delta_k^* = \gamma \Delta_k^{\min}$ for a given $0 < \gamma \leq 1$, and go to step (iii).

(b) If $T_k^+ < \Delta_k^{\min}$, then set $t_{k+1} = t_k + T_k^+$ and go to step (iv).

(iii) The ETS provides the next update time $t_{k+1}(> t_k)$ in the following way:

(a) Set $m = 1$.

(b) At an evaluation time $t = t_k + m\delta_k^*$, $m \in \mathbb{N}_{\geq 1}$, measure the state $x(t)$, and check the event-triggered conditions given by (24), (25).

(c) If (24) and (25) are both satisfied, then apply $u^*(\xi)$ for $\xi \in [t, t + \delta_k^*)$. Then, set $m \leftarrow m + 1$ and go back to step (b). Otherwise, set $t_{k+1} = t$ and go to step (iv).

(iv) $k \leftarrow k + 1$ and go back to step (i).

Remark 1 (On tuning the parameter $\gamma$) If $\gamma$ is chosen larger, then we obtain larger $\delta_k^* = \gamma \Delta_k^{\min}$, and thus a smaller number of state measurements and evaluations may be obtained. However, due to the increased value of $\delta_k^*$, the right hand side of (24) becomes smaller and the event-triggered condition becomes more conservative, resulting in a larger number of OCPs. Therefore, there exists a trade-off between the number of OCPs and the number of state measurements, and the parameter $\gamma$ plays an important role to regulate this trade-off. This property serves as one of the benefits of our proposed strategy, as we can now appropriately select $\gamma$ according to whether we would like to focus on reducing the number of OCPs or number of state measurements.

4.1 Self-triggered strategy

In the self-triggered strategy, the next update time $t_{k+1}$ is pre-determined at $t_k$ as soon as the OCP is solved, without having to evaluate the event-triggered condition. To obtain the self-triggered strategy, recall that the minimum inter-event time of satisfying (18) and (19) is $\min\{\Delta_k^{\min}, T_k^+\}$, where $\Delta_k^{\min}$ is given by (21). Since $\Delta_k^{\min}, T_k^+$ can be obtained at $t_k$ (immediately after solving the OCP), we can simply set $t_{k+1}$ as

$$t_{k+1} = t_k + \min\{\Delta_k^{\min}, T_k^+\}, \quad (26)$$

Although considering the minimum inter-event time may lead to more conservative result than the previous even-triggered strategy, the evaluations of the event-triggered condition and the state measurements are no longer required between two update times of the OCP.

The following self-triggered strategy is now obtained:

Algorithm 2 (Self-triggered Strategy):

(i) For any update times $t_k, k \in \mathbb{N}_{\geq 0}$, if $x(t_k) \in \Phi$, then switch to the local controller $\kappa(x)$ as a dual mode strategy. Otherwise, solve Problem 1 and obtain the optimal control $u^*(\xi)$ and the corresponding state trajectory $\hat{x}^*(\xi)$ for all $\xi \in [t_k, t_k + T_k]$. Then, calculate $T_k^+$ as the time interval when the state reaches $\Phi_f$, i.e., $\hat{x}^*(t_k + T_k^+) \in \partial \Phi_f$. Furthermore, calculate $\Delta_k^{\min}$ according to (21).

(ii) The SDS sets the next update time $t_{k+1}$ as

$$t_{k+1} = t_k + \min\{\Delta_k^{\min}, T_k^+\} \quad (27)$$

and applies $u^*(t)$ for all $t \in [t_k, t_{k+1})$.

(iii) $k \leftarrow k + 1$ and go back to step (i).

5 Stability

In this section we show the stability of the closed loop system. For a given initial prediction horizon $T_0 > 0$, we let $X(T_0)$ be the set of states such that a feasible solution to Problem 1 exists. We will prove in the following that, any state trajectories starting from inside $X(T_0)$ will eventually enter $\Phi$ within a prescribed finite time interval.

Theorem 2 Consider the nonlinear system given by (1), and suppose that Algorithm 1 or Algorithm 2 is implemented. Then, for any realization of $w(t)$ satisfying $\|w(t)\|_{P_2} \leq \min\{\bar{w}_{\text{max}}, \hat{w}_{\text{max}}\}, \forall t \geq t_0$, where $\bar{w}_{\text{max}}$ and $\hat{w}_{\text{max}}$ are given by (7), (14) respectively, any state trajectories starting from $x(t_0) \in X(T_0)$ enter $\Phi$ within the time interval $T_0^+/\alpha$, and remain in $\Phi$ for all the future times.

PROOF. We prove the statement by contradiction. Assume at $t_k$ that we have $t_k - t_0 > T_0^+/\alpha$, and $x(t_k)$ is outside of $\Phi$, i.e., $x(t_k) \notin \Phi$. Since $x(t_k) \notin \Phi$ and $\Phi_f \subset \Phi$, we have $T_k^+ > 0$. As $x(t_0) \in X(T_0)$ and $\|w(t)\|_{P_2} \leq \hat{w}_{\text{max}}, \forall t \geq t_0$, applying Algorithm 1 or Algorithm 2 ensures that the feasibility is guaranteed for all $t_0, t_1, \ldots, t_k$. 

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Thus, we recursively obtain from (10) that:
\[
T_k^* \leq T_{k-1}^* - \alpha \Delta_k - 1 \leq T_{k-2}^* - \alpha (\Delta_k - 1 + \Delta_k - 2)
\]
\[
\vdots \leq T_0^* - \alpha \sum_{l=1}^{k-1} \Delta_l
\]
\[
= T_0^* - \alpha (t_k - t_{k-1} + t_{k-2} + \cdots + t - t_0)
\]
\[
= T_0^* - \alpha (t_k - t_0).
\]
Thus, by the assumption \( t_k - t_0 \geq T_0^*/\alpha \), we obtain \( T_k^* \leq 0 \). However, this clearly contradicts to the fact that we have \( T_0^* > 0 \). Thus, it is shown that the state enters \( \Phi \) within the time interval \( T_0^*/\alpha \). Furthermore, since from Lemma 1, \( \Phi \) is a positively invariant set with the disturbance satisfying \( ||w(t)|| \leq \tilde{w}_{\text{max}} \), the state remains in \( \Phi \) for all future times. This completes the proof.

**Remark 2** Aside from the event-triggered strategy, one of the important results of this paper is that, by guaranteeing stability without using optimal cost, the maximum time of convergence is explicitly obtained by Theorem 2. Although the convergence time has been analyzed for linear discrete-time systems, e.g., \([22]\), this paper derives it for nonlinear continuous-time systems with additive bounded disturbances. \( \square \)

**Remark 3 (Convergence time v.s. Disturbance)**
If \( \alpha \) is chosen larger, then we obtain smaller \( T_0^*/\alpha \) and faster convergence is obtained. However, this in turn means from (14) that the allowable size of disturbance becomes smaller, which implies that the robustness to the noise or model uncertainty may be degraded. Therefore, there exists a trade-off between the convergence time of the state trajectory and the allowable size of the disturbance, and this trade-off can be regulated by tuning \( \alpha \). \( \square \)

6 Simulation Results

As a simulation example, we consider the spring-mass system; the state vector \( x = [x_1; x_2] \in \mathbb{R}^2 \) consists of the position of the object \( x_1 \) and its velocity \( x_2 \). The dynamics are given by
\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u + w, \quad (28)
\]

where \( k = 2 \) is the spring coefficient and \( m = 1 \) is the mass. The matrices for the stage cost are \( Q = 0.05 I_2 \), \( R = 0.01 \), and the initial prediction horizon is set to \( T_0 = 8.0 \). To obtain the parameter \( \varepsilon \) and the local controller \( \kappa \), we follow the steps in \([23]\), resulting in \( \varepsilon = 0.02 \) and \( \kappa = Kx \) with \( K = [1.0 \, 2.6] \). We obtain the matrix \( P_f \) as
\[
P_f = \begin{bmatrix} 0.079 & 0.010 \\ 0.010 & 0.26 \end{bmatrix}, \quad (29)
\]

Fig. 6 represents state trajectories under Algorithm 1 (event-triggered strategy) with \( \gamma = 1.0 \) and Algorithm 2 (self-triggered strategy). From the figure, state trajectories converge to a region around the origin under both algorithms. The resulting convergence times needed for the state to enter \( \Phi \), are illustrated in Table 1 with the upper bound of convergence \( T_0^*/\alpha \) according to Theorem 2. From Table 1, it is shown that the state trajectories under both cases converge to the local set \( \Phi \) within \( T_0^*/\alpha \).

The computed Lipschitz constant is given by \( L_f = 0.11 \) and we set \( \varepsilon_f = 0.01, \alpha = 0.8 \). From Theorem 1, the feasibility is guaranteed if \( \tilde{w}_{\text{max}} = 4.2 \times 10^{-4} \) and from Lemma 1 the region \( \Phi \) is positively invariant if \( \tilde{w}_{\text{max}} = 2.0 \times 10^{-2} \). Taking into account both restrictions, we assume that the additive disturbance satisfies \( ||w||_{P_f} \leq 4.2 \times 10^{-4} \).

Fig.7 shows the inter-event times of solving the OCPs under Algorithm 1 (\( \gamma = 0.2, 1.0 \)), Algorithm 2, and the preliminary approach presented in \([21]\). From Fig. 7, the proposed approach attains longer inter-event times as the state becomes closer to the origin. This is due to the fact that the time interval \( T_k^* \) becomes smaller as the state approaches \( \Phi_f \), and the triggering condition (24) becomes more likely to be satisfied. We can see from the figure that the proposed schemes achieve less conservative results than our previous approach. This may be because the previous event-triggered conditions include Lipschitz constant parameters for the stage and terminal cost (see \([21]\)), which may be a potential source of conservativeness.
Fig. 7. Inter-event times of solving OCPs under Algorithm 1 \((\gamma = 0.2, 1)\), Algorithm 2 (self-triggered MPC), and the preliminary approach presented in [21].

![Graph showing inter-event times of solving OCPs]

In Remark 1, we point out that there exists a trade-off between the number of OCPs and the number of state measurements. In order to illustrate this, we have plotted in Fig. 8 the average number of OCPs to be solved and the average number of state measurements until the state converges \(\Phi\), under various values of \(\gamma\) (ranging from \(\gamma = 0.01\) to \(\gamma = 1\)). For each \(\gamma\), we conducted the simulation for 100 times to obtain the average values. We can see from the lower figure of Fig. 8, that the number of state measurements becomes smaller as \(\gamma\) is selected larger. However, this in turn leads from the upper figure that a larger number of OCPs needs to be solved. This is due to the fact that the event-triggered condition (24) becomes more conservative as larger \(\delta_k\) is selected. Thus, it is shown that there exists a trade-off between the number of OCPs and the number of state measurements, and it can be regulated through the tuning of \(\gamma\).

Fig. 8. Average number of OCPs (upper) and state measurements (lower) until the state converges \(\Phi\).

![Graph showing average number of OCPs and state measurements]

7 Conclusions

In this paper, we propose a new event-triggered strategy of MPC for nonlinear continuous-time systems. The proposed method is derived based on the new feasibility and stability results by imposing the terminal constraint with an adaptive prediction horizon. In the derivations of stability, we evaluate the time interval when the optimal state trajectory enters the local set \(\Phi_f\), and it is shown that the state converges \(\Phi\) within a prescribed finite time interval. Furthermore, the proposed event-triggered conditions are evaluated only at certain sampling time intervals, aiming at reducing both sensing and computational load. A simulation example illustrates the effectiveness of the proposed scheme.

A Proof of Lemma 1

Consider a linearization of (1) around the origin for the non-disturbance case:

\[
\dot{x} = A_f x + B_f u, \quad (A.1)
\]

where \(A_f = \partial f/\partial x(0,0)\) and \(B_f = \partial f/\partial u(0,0)\). Since \((A.1)\) is stabilizable from Assumption 1, we can find a state feedback controller \(\kappa(x) = K x\) such that \(A_c = A_f + B_f K\) is Hurwitz and the closed loop system \(\dot{x} = A_c x\) is thus asymptotically stable. Choose a matrix \(P\) such that the following Lyapunov equation holds: \(PA_c + A_c^T P = -(Q + K^T R K)\) where \(Q\) and \(R\) are matrices for the stage cost defined in (4). Then, the time derivative of the function \(V_f = x^T P x\) along a trajectory of the nonlinear system \(\dot{x} = f(x, \kappa(x))\) yields:

\[
\dot{V}_f(x) = -x^T (Q + K^T R K) x + 2x^T P \phi(x) \\
\leq -x^T (Q + K^T R K) x \left( 1 - 2\lambda_{\max}(\bar{Q}_P) \frac{||\phi(x)||_P}{||x||_P} \right), \quad (A.2)
\]

where \(\phi(x) = f(x, \kappa(x)) - A_c x, \) and \(Q_P = P^{-1/2}(Q + K^T R K)P^{-1/2}.\) Since \(||\phi(x)||_P/||x||_P \to 0\) as \(||x||_P \to 0\), there exists a positive constant \(0 < \varepsilon_0 < \infty\) such that \(||\phi(x)||_P/||x||_P \leq 1/(4\lambda_{\max}(\bar{Q}_P))\) for \(||x||_P \leq \varepsilon_0.\) Let \(0 < \varepsilon \leq \varepsilon_0\) such that for all \(||x||_P \leq \varepsilon, \kappa(x) = K x \in U.\) By letting \(\Phi = \{x \in \mathbb{R}^n \mid V_f(x) \leq \varepsilon^2\},\) and from (A.2), we obtain:

\[
\dot{V}_f(x) \leq -\frac{1}{2} x^T (Q + K^T R K) x, \quad (A.3)
\]

for all \(x \in \Phi.\) This completes the proof of (6).

Now consider the time derivative of the function \(V_f\) along a trajectory of the nonlinear system with additive dis-
turbances $\dot{x} = f(x, \kappa(x)) + w$:
\[
\dot{V}_f(x) = -x^T(Q + K^T R K)x + 2x^T P \phi(x) + 2x^T P w = \leq -x^T(Q + K^T R K)x \left(1 - 2 \lambda_{\text{max}}(P) \right) \left| \phi(x) \right|_{P} - 2 \lambda_{\text{max}}(P) \left| w \right|_{P},
\]
and consider also a compact set as a boundary of $\Phi$; $\partial \Phi = \{ x \in \mathbb{R}^n \mid V_f(x) = \varepsilon^2 \}$. From (A.4), we obtain $\dot{V}_f \leq 0$ for $x \in \partial \Phi$, if $\left| w \right|_{P} \leq \varepsilon/(4 \lambda_{\text{max}}(P))$. Thus, $\Phi$ is a positive invariant set for the closed loop system $\dot{x} = f(x, \kappa(x)) + w$ if the disturbance satisfies $\left| w \right|_{P} \leq \varepsilon/(4 \lambda_{\text{max}}(P))$. This completes the proof of Lemma 1.

References


