Quiver gauge theory and quiver W-algebra

箙ゲージ理論と箙W代数

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Abstract

We provide a brief introduction to quiver W-algebra, which is a gauge theory construction of W-algebra. We show that the gauge theory partition function is generated by the screening charge, and the generating current of the W-algebra is given by the qq-character, a double quantization of the character for the fundamental representations associated with the quiver.

1. Introduction and summary

The Virasoro algebra, and the W-algebra in general, is a symmetry algebra describing the infinite dimensional conformal symmetry appearing in several fields of physics, e.g., string theory, critical phenomena in statistical mechanics, mathematical physics. The q-deformation of Virasoro/W-algebra was introduced in the middle of 1990s [1, 2], but its physical realization, namely a physical system for which the q-Virasoro/W-algebra plays a role as the symmetry algebra, has not been found for a long time. In the late 2000s, a new physical realization of conformal algebra was proposed, called the AGT relation [3], which shows a connection between 4d supersymmetric gauge theory and 2d conformal field theory. In this case, the conformal algebra is naturally q-deformed by considering 5d gauge theory compactified on a circle $\mathbb{R}^4 \times S^1$ [4].

Another realization of the q-conformal algebra is quiver W-algebra [5, 6, 7], which relates Γ-quiver gauge theory in 5d to the q-deformed algebra $W_{q_1,q_2}(\Gamma)$, while the AGT relation states a connection between G-gauge symmetric theory and W-algebra $W_{q_1,q_2}(G)$. Indeed a duality exchanges Γ and G, and explains a relation of these two connections with the conformal algebra. The formalism of quiver W-algebra gives rise to several new features as follows:

Affine and hyperbolic W-algebras [5]

For finite type simply-laced quiver, $\Gamma = ADE$, the algebra $W_{q_1,q_2}(\Gamma)$ reproduces the construction by Frenkel–Reshetikhin [8]. If we start with non-finite type quiver, namely affine or hyperbolic quiver, we obtain a new family of W-algebras, associated with affine/hyperbolic Lie algebra.

Elliptic deformation of W-algebras [6]

The q-deformed algebra arises from 5d gauge theory on $\mathbb{R}^4 \times S^1$. Applying our formalism to 6d gauge theory compactified on a torus $\mathbb{R}^4 \times T^2$, we obtain the elliptic deformation of W-algebras.

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Fractional quiver gauge theory [7]

Considering a quiver consisting of vertices and edges, inevitably it turns out to be simply-laced. Utilizing a connection between gauge theory and W-algebras, we define the fractional quiver gauge theory, reproducing the W-algebras associated to non-simply-laced Lie algebras, which also implies fractionalization of quiver variety.

In this article, we explain basic aspects of quiver W-algebra, including the operator-valued gauge theory partition function (Z-state), and the construction of the generating current for the algebra, which is given by the double quantization of characters associated with quiver (qq-character.) Please refer to the original papers [5, 6, 7] for details. See also a review article [9] and the shortened version in Japanese [10].

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2. Quiver gauge theory

Gauge theory is a quantum field theory, which owes its dynamics to a gauge field $A \in \mathfrak{g}$, a Lie algebra valued one-form field defined on a spacetime. It has a symmetry under the gauge transformation $A \to gAg^{-1} + gdg^{-1}$ with $g \in G$ (adjoint representation), where G is the gauge group, the Lie group associated with the algebra $\mathfrak{g}$.

Quiver gauge theory, in general, has several gauge fields transforming under the corresponding multiple gauge groups, characterized by a quiver graph Γ.[1] Let Γ be a quiver with a set of nodes (vertices) Γ_0 and arrows (directed edges) Γ_1. An edge from the node i to j is denoted by $e : i \to j$. We assign a gauge group $\mathrm{U}(n_i)$ to each node $i \in \Gamma_0$ under which a gauge field A_i transforms in adjoint representation, $A_i \to g_i A_i g_i^{-1} + g_i dg_i^{-1}$ with $g_i \in \mathrm{U}(n_i)$. For each edge $e : i \to j$ we assign a field transforms in bifundamental representation of $\mathrm{U}(n_i)$ and $\mathrm{U}(n_j)$, namely $(\bar{\mathbf{n}}_i, \mathbf{n}_j)$.

2.1. Equivariant localization

We consider four-dimensional Euclidean spacetime $\mathbb{R}^4 = \mathbb{C}^2$ for gauge theory. We define the field strength (curvature; two-form) from the gauge field $F_i = dA_i + A_i^2$ for the node $i \in \Gamma_0$. We are in particular interested in the instanton (anti-self-dual; ASD) configuration, $*F_i = -F_i$, which can be a solution to the classical e.o.m (Yang–Mills equation). Such a configuration is characterized by instanton (2nd Chern) numbers:

$$-\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\, F_i^2 = k_i \,. \tag{1}$$

We define a set of gauge group ranks $n = (n_i)_{i\in\Gamma_0}$ and instanton numbers $k = (k_i)_{i\in\Gamma_0}$. We denote the instanton moduli space by $\mathcal{M}_{n,k}$ described using the ADHM construction. Then we define the gauge theory partition function

$$Z = \sum_k \mathfrak{q}^k \int_{\mathcal{M}_{n,k}} 1 \tag{2}$$

[1] We basically follow the notation of [5, 6, 7].

where $\mathfrak{q}^k = \prod_{i\in\Gamma_0} \mathfrak{q}_i^{k_i}$ with $\mathfrak{q}_i = \exp\left(2\pi\sqrt{-1}\tau_i\right)$, and $(\tau_i)_{i\in\Gamma_0}$ is the complexified gauge coupling constant. We remark that this simplified partition function can be derived using the path integral formalism with extended supersymmetry.

Evaluation of the integral over the instanton moduli space can be performed by applying the equivariant K-theoretic localization (See, for example, [11]): The integral is localized on discrete fixed points under the torus action, given by $(\nu_{i,\alpha})_{\alpha=1,\ldots,n_i} := (e^{a_{i,\alpha}})_{\alpha=1,\ldots,n_i} \in \mathrm{U}(1)^{n_i} \subset \mathrm{U}(n_i)$ for gauge symmetry, and $(q_1, q_2) := (e^{\epsilon_1}, e^{\epsilon_2}) \in \mathrm{U}(1)^2 \subset \mathrm{SO}(4)$ for spacetime rotation symmetry. We define $q = q_1 q_2 = e^{\epsilon_1+\epsilon_2}$. The fixed point is labeled by the multiple partition $(\lambda_{i,\alpha})_{i\in\Gamma_0,\alpha=1,\ldots,n_i}$, satisfying the non-increasing condition $\lambda_{i,\alpha,1} \ge \lambda_{i,\alpha,2} \ge \cdots \ge 0 = \cdots = 0$. We have two vector bundles, whose Chern characters are given by

$$N_i = \sum_{\alpha=1}^{n_i} \nu_{i,\alpha}\,, \qquad K_i = \sum_{\alpha=1}^{n_i} \sum_{(s_1,s_2)\in\lambda_{i,\alpha}} q_1^{s_1-1} q_2^{s_2-1} \nu_{i,\alpha}\,. \tag{3}$$

Then we define (the character of) the universal sheaves from these two bundles evaluated at the fixed point $(\lambda_{i,\alpha})$

$$\mathbf{Y}_i = N_i - (1-q_1)(1-q_2)K_i = \begin{cases} (1-q_1) \displaystyle\sum_{x\in\mathcal{X}_i} x \\ (1-q_2) \displaystyle\sum_{x\in\mathcal{X}_i^{\mathrm{T}}} x \end{cases}, \tag{4}$$

where we define

$$\mathcal{X}_i = \left\{ x_{i,\alpha,k} = q_1^{k-1} q_2^{\lambda_{i,\alpha,k}} \nu_{i,\alpha} \right\}_{\alpha=1,\ldots,n_i,\, k=1,\ldots,\infty}, \qquad \mathcal{X} = \bigsqcup_{i\in\Gamma_0} \mathcal{X}_i \tag{5a}$$

$$\mathcal{X}_i^{\mathrm{T}} = \left\{ x_{i,\alpha,k}^{\mathrm{T}} = q_1^{\lambda_{i,\alpha,k}^{\mathrm{T}}} q_2^{k-1} \nu_{i,\alpha} \right\}_{\alpha=1,\ldots,n_i,\, k=1,\ldots,\infty}, \qquad \mathcal{X}^{\mathrm{T}} = \bigsqcup_{i\in\Gamma_0} \mathcal{X}_i^{\mathrm{T}} \tag{5b}$$

with the transposed partition denoted by $(\lambda_{i,\alpha}^{\mathrm{T}})_{i\in\Gamma_0,\alpha=1,\ldots,n_i}$. The map $(\lambda_{i,\alpha,k}) \to (x_{i,\alpha,k})$ corresponds to that from the Young diagram to the Maya diagram. We remark that the first expression is manifestly symmetric under exchange $q_1 \leftrightarrow q_2$, but it becomes not manifest in the second one.

We define the degree-m character of the universal sheaf, obtained through the Adams operation,

$$\mathbf{Y}_i^{[m]} = (1-q_1^m) \sum_{x\in\mathcal{X}_i} x^m\,, \tag{6}$$

which is given by a degree-m power sum symmetric polynomial of infinitely many variables. This is called the gauge invariant; single trace; chiral ring; observable in the context of supersymmetric gauge theory. Then we define the t-extended partition function [11, 12]

$$Z(t) = \sum_k \mathfrak{q}^k \int_{\mathcal{M}_{n,k}} \exp\left(\sum_{i\in\Gamma_0} \sum_{m=1}^{\infty} t_{i,m} \mathbf{Y}_i^{[m]} \right), \tag{7}$$

which plays a role as the generating function: The derivative with the conjugate t-variable gives rise to the expectation value of the observable

$$\left\langle \mathbf{Y}_i^{[m]} \right\rangle = \frac{\partial}{\partial t_{i,m}} \log Z(t) \Big|_{t\to 0} . \tag{8}$$

This operator average is taken with respect to the plain partition function (2) given by $Z(t=0) = Z$.

3. Z-state

Since, as mentioned above, the universal sheaf character is given by the power sum polynomial with infinite variables, we can consider the operator formalism using the free field realization. (See, for example, [13] and also [14].) Namely, identifying the derivative with the t-variable with the observable, $t_{i,-m} := \partial / \partial t_{i,m} \leftrightarrow \mathbf{Y}_i^{[m]}$, we obtain the Heisenberg algebra, $[t_{i,-m}, t_{j,m'}] = \delta_{i,j}\delta_{m,m'}$. In this sense, the t-variable is promoted to the operator generating the Fock space $\mathcal{F} = \mathbb{C}[[t_{i,1}, t_{i,2}, \cdots]]|0\rangle$ with the vacuum state annihilated by any negative modes $t_{i,-m}|0\rangle = 0$ for $m > 0$. Thus, the t-extended partition function, depending on the t-variables, is also promoted to an operator. From this point of view, we define the Z-state through the operator/state correspondence,

$$|Z\rangle = Z(t)|0\rangle . \tag{9}$$

It has been shown that the Z-state for Γ-quiver gauge theory compactified on a circle $\mathbb{R}^4 \times S^1$ is generated by the screening charge associated with quiver W-algebra $W_{q_1,q_2}(\Gamma)$:

Z-state [5]

Let $\mathsf{i} : \mathcal{X} \to \Gamma_0$, such that $\mathsf{i}(x) = i$ for $x \in \mathcal{X}_i$. Then the Z-state is generated by screening charges of the algebra $W_{q_1,q_2}(\Gamma)$,

$$|Z\rangle = \prod_{x \in \mathring{\mathcal{X}}}^{\succ} \mathsf{S}_{\mathsf{i}(x),x} |0\rangle . \tag{10}$$

The configuration $\mathring{\mathcal{X}}$ is defined with the empty configuration $(\lambda_{i,\alpha}) = \emptyset$,

$$\mathring{\mathcal{X}}_i = \{\mathring{x}_{i,\alpha,k} = q_1^{k-1} \nu_{i,\alpha}\}_{\alpha=1,\ldots,n_i, k=1,\ldots,\infty} , \quad \mathring{\mathcal{X}} = \bigsqcup_{i\in\Gamma_0} \mathring{\mathcal{X}}_i . \tag{11}$$

The screening charge is defined as a discrete sum (could be formulated using Jackson integral) of the screening current

$$\mathsf{S}_{i,x} = \sum_{k\in\mathbb{Z}} S_{i,q_2^k x} \tag{12}$$

with the free field realization

$$S_{i,x} = : \exp\left(s_{i,0} \log s + \tilde{s}_{i,0} + \sum_{m\neq 0} s_{i,m}\, x^{-m} \right) : \tag{13}$$

and the commutation relations

$$\left[s_{i,m},\, s_{j,m'}\right] = -\frac{1}{m}\frac{1-q_1^m}{1-q_2^{-m}}\, c_{ji}^{[m]}\delta_{m+m',0}\,, \qquad \left[\tilde{s}_{i,0},\, s_{j,m}\right] = -\beta\delta_{0,m}c_{jk}^{[0]}\,, \quad \beta = -\frac{\epsilon_1}{\epsilon_2}\,. \tag{14}$$

The matrix $(c_{ij}^{[m]})$ is the mass-deformed q-Cartan matrix, which is reduced to the ordinary quiver Cartan matrix in the limit $m \to 0$,

$$\begin{aligned} c_{ij}^{[m]} &= (1+q^{-m})\delta_{ij} - \sum_{e:i\to j}\mu_e^{-m} - \sum_{e:j\to i}\mu_e q^{-m} \\ &\xrightarrow{m\to 0}\ 2\delta_{ij} - \#(e:i\to j) - \#(e:j\to i) \end{aligned} \tag{15}$$

where $(\mu_e)_{e\in\Gamma_1}$ is the multiplicative bifundamental mass parameter. The mass deformation plays an essential role to define the algebra, in particular, associated to the affine quiver. See Sec. 5. We remark the transposition $c_{ji}^{[m]} = q^{-m}c_{ij}^{[-m]}$.

The Z-state (10) is responsible for the vector and bifundamental hypermultiplets. To obtain the (anti)fundamental hypermultiplet contribution, we insert additional vertex operators

$$|Z\rangle = \left(\prod_{x\in\mathcal{X}_\mathrm{f}} \mathsf{V}_{\mathsf{i}(x),x}\right)\left(\prod_{x\in\mathring{\mathcal{X}}}^{\succ} \mathsf{S}_{\mathsf{i}(x),x}\right)\left(\prod_{x\in\mathcal{X}_\mathrm{af}} \mathsf{V}_{\mathsf{i}(x),x}\right)|0\rangle \tag{16}$$

where $\mathcal{X}_\mathrm{f} = \{\mu_{i,f}\}_{i\in\Gamma_0, f=1,\ldots,n_i^\mathrm{f}}$ and $\mathcal{X}_\mathrm{af} = \{\tilde{\mu}_{i,f}\}_{i\in\Gamma_0, f=1,\ldots,n_i^\mathrm{af}}$ are sets of the multiplicative fundamental and antifundamental mass parameters, obeying the OPE

$$\mathsf{V}_{i,x}S_{i,x'} = \left(\frac{x'}{x};q_2\right)_\infty^{-1} :\mathsf{V}_{i,x}S_{i,x'}:\,, \qquad S_{i,x'}\mathsf{V}_{i,x} = \left(q_2\frac{x}{x'};q_2\right)_\infty :\mathsf{V}_{i,x}S_{i,x'}:\,. \tag{17}$$

The q-Pochhammer symbol is defined as $(z;q)_n = \prod_{m=0}^{n-1}(1-zq^m)$.

Since the dual vacuum obeys $\langle 0|t_{i,m} = 0$ for $m > 0$, the plain partition function given by imposing $t = 0$ is correspondingly given as a correlator

$$Z(t=0) = \langle 0|Z(t)|0\rangle\,. \tag{18}$$

Such a correlator realization of the partition function (18) resembles the AGT relation [3], which states that the gauge theory partition function with gauge group G is given by a conformal block of $W(G)$ algebra, while quiver W-algebra is sensitive to quiver structure, but not to G. The relation between these two descriptions is understood as a base/fibre (geometry); spectral (integrable system); S-duality (string theory). We also remark that the expression (18) immediately leads to the discretized version of the Dotsenko–Fateev integral

$$Z(t=0) = \sum_{\mathcal{X}} Z_{\mathcal{X}}(t=0) = \sum_{\mathcal{X}}\langle 0|\left(\prod_{x\in\mathcal{X}_\mathrm{f}} \mathsf{V}_{\mathsf{i}(x),x}\right)\left(\prod_{x\in\mathcal{X}}^{\succ} S_{\mathsf{i}(x),x}\right)\left(\prod_{x\in\mathcal{X}_\mathrm{af}} \mathsf{V}_{\mathsf{i}(x),x}\right)|0\rangle\,. \tag{19}$$

Namely, each contribution from the fixed point configuration $\mathcal{X}$ is given by a correlator of the screening currents with the vertex operators. In other words, the screening current generates the contribution associated with a specific configuration

$$|Z_{\mathcal{X}}\rangle = \left(\prod_{x\in\mathcal{X}_{\mathrm{f}}} \mathsf{V}_{\mathsf{i}(x),x} \right) \left(\prod_{x\in\mathcal{X}}^{\succ} S_{\mathsf{i}(x),x} \right) \left(\prod_{x\in\mathcal{X}_{\mathrm{af}}} \mathsf{V}_{\mathsf{i}(x),x} \right) |0\rangle \,. \tag{20}$$

Although this correlator involves infinitely many operators, one can truncate the number of screening charges by considering the codimension-2 defect. See, for example, [15].

4. Quiver W-algebra

We define another vertex operator, called the Y-operator:

$$\mathsf{Y}_{i,x} = \; q_1^{\tilde{\rho}_i} : \exp\left(y_{i,0} + \sum_{m\neq 0} y_{i,m}\, x^{-m} \right) : \tag{21}$$

where $(\tilde{\rho})_{i\in\Gamma_0}$ is the Weyl vector in the simple root basis (as long as $\det(c_{ij}^{[0]}) \neq 0$), and the commutation relations are defined

$$\Big[y_{i,m},\, s_{j,m'} \Big] = -\frac{1}{m}(1-q_1^m)\delta_{i,j}\delta_{m+m',0}\,, \qquad \Big[\tilde{s}_{i,0}, y_{j,0} \Big] = -\delta_{ij}\log q_1\,. \tag{22}$$

The OPE of Y-operator and the screening current is then given by

$$\mathsf{Y}_{i,x} S_{j,x'} = \frac{1-x'/x}{1-q_1 x'/x} : \mathsf{Y}_{i,x} S_{j,x'} : , \quad S_{j,x'}\mathsf{Y}_{i,x} = q_1^{-1}\frac{1-x/x'}{1-q_1^{-1}x/x'} : \mathsf{Y}_{i,x} S_{j,x'} : \quad (i=j) \tag{23}$$

while this OPE becomes trivial if $i \neq j$. It gives rise to the commutation relation

$$\Big[\mathsf{Y}_{i,x},\, S_{j,x'} \Big] = \left(1-q_1^{-1}\right)\delta\left(q_1\frac{x'}{x}\right) : \mathsf{Y}_{i,x} S_{j,x'} : \tag{24}$$

where the multiplicative δ-function is defined $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$. The Y-operator average is computed using the OPE factor as follows:

$$\langle 0|\mathsf{Y}_{i,x}|Z_{\mathcal{X}}\rangle = q_1^{\tilde{\rho}_i} \prod_{x\in\mathcal{X}_i} \frac{1-x'/x}{1-q_1 x'/x}\langle 0|Z_{\mathcal{X}}\rangle \tag{25}$$

Since we have

$$\prod_{x\in\mathcal{X}_i} \frac{1-x'/x}{1-q_1 x'/x} = \exp\left(\sum_{m=1}^{\infty} -\frac{x^{-m}}{m}\mathbf{Y}_i^{[m]} \right) , \tag{26}$$

the Y-operator average plays a role of the generating function of the observable (8).

One can construct the generators of W-algebras using this Y-operator (the q-Sugawara construction): It has been shown in [5] that the generating currents of quiver W-algebra $W_{q_1,q_2}(\Gamma)$ is given by the operator-valued qq-character associated with quiver Γ, which turns out to be a commutant of the screening charge. We define the iWeyl reflection incorporated by the A-operator

$$\mathsf{Y}_{i,x} \;\longrightarrow\; \mathsf{Y}_{i,x}\,\mathsf{A}_{i,q^{-1}}^{-1} \tag{27}$$

with the definition

$$\mathsf{A}_{i,x} = q_1 : \exp\left(a_{i,0} + \sum_{m\neq 0} a_{i,m}\, x^{-m}\right) : \tag{28}$$

where

$$a_{i,m} = \sum_{j\in\Gamma_0} y_{j,m}\, c_{ji}^{[m]} \,. \tag{29}$$

Thus the A-operator is written in terms of the Y-operators. Let us write down the commutation relations for the free fields:

$$\left[y_{i,m},\, y_{j,m'}\right] = -\frac{1}{m}(1-q_1^m)(1-q_2^m)\, \tilde{c}_{ij}^{[-m]}\, \delta_{m+m',0}\,, \tag{30}$$

$$\left[a_{i,m},\, a_{j,m'}\right] = -\frac{1}{m}(1-q_1^m)(1-q_2^m)\, c_{ji}^{[m]}\, \delta_{m+m',0}\,. \tag{31}$$

where $(\tilde{c}_{ij}^{[m]})$ is the inverse of the q-Cartan matrix (15). We remark that, in the limit $q_1(q_2) \to 1$, these commutation relations become trivial due to the factor $(1-q_1^n)(1-q_2^n)$, which implies the quantum algebra becomes the classical commutative algebra: It still holds the Poisson structure even in this limit.

The Y and A operators play roles of the weight and root vectors: The Weyl reflection is generated by the root vector. Then the qq-character is given by

$$\mathsf{T}_{i,x} = \mathsf{Y}_{i,x} + \mathsf{Y}_{i,x}\, \mathsf{A}_{i,q^{-1}}^{-1} + \cdots \,. \tag{32}$$

Monomials generated by the reflection may include the Y-operators both in numerator and denominator. We apply the iWeyl reflection as long as the Y-operator appears in the numerator, which terminates within finite processes for finite type quiver $\Gamma = ADE$, while it becomes an infinite series for affine/hyperbolic quiver. We remark that the qq-character has the integral formula from the quiver variety associated with the quiver Γ. See [16] for details.

Although the Y-operator does not commute with the screening charge as shown in (24), the qq-character $\mathsf{T}_{i,x}$ commutes with the screening charge

$$\left[\mathsf{T}_{i,x}, \mathsf{S}_{j,x'}\right] = 0\,, \tag{33}$$

which implies the following:

The operator-valued qq-character provides the free field realization of the generating current of the algebra $W_{q_1,q_2}(\Gamma)$, which is a commutant of the associated screening charge.

Let us demonstrate this statement with an example.

4.1. A_1 quiver

Let us consider the simplest example: A_1 quiver, consisting of a single node without any edges. In this case, the fundamental qq-character is given by

$$\mathsf{T}_{1,x} = \mathsf{Y}_{1,x} + \mathsf{Y}_{i,q^{-1}x}^{-1} \tag{34}$$

which turns out to be the generating current of the q-deformed Virasoro algebra [2]. Then, in the classical limit, it is reduced to the fundamental representation character of SU(2).

We can also consider higher representations of SU(2). The spin-$\frac{\ell}{2}$ ((2ℓ+1)-dimensional) representation character is given by $\chi_\ell = y^\ell + y^{\ell-1} + \cdots + y^{-\ell}$. Then the corresponding qq-character is generated by the product of the Y-operators with ℓ arguments $\mathsf{w} = (w_f)_{f=1,\ldots,\ell}$,

$$\begin{aligned} \mathsf{T}_{1,\mathsf{w}} &= : \mathsf{Y}_{1,w_1} \cdots \mathsf{Y}_{1,w_\ell} : + \cdots \\ &= \sum_{I \cup J = \{1,\ldots,\ell\}} \prod_{i \in I, j \in J} \mathcal{S}\left(\frac{w_i}{w_j}\right) : \prod_{i \in I} \mathsf{Y}_{1,w_i} \mathsf{Y}^{-1}_{1,q^{-1}w_j} : . \end{aligned} \tag{35}$$

where

$$\mathcal{S}(z) = \frac{(1-q_1 x)(1-q_2 x)}{(1-x)(1-qx)} . \tag{36}$$

We remark that this qq-character contains 2^ℓ terms, so that one cannot see decomposition into the irreducible representations of SU(2). Nevertheless, the q-character [17] of the corresponding representation is obtained from the qq-character in the limit $(w_f) \to (q_1^{f-1} w)$, and then taking $q_2 \to 1$, since $\mathcal{S}(q_1^{-1}) = 0$ and $\mathcal{S}(q_1^n) \overset{q_2 \to 1}{\to} 1$ for $n \in \mathbb{Z} \backslash \{-1, 0\}$.

In particular, the case with $\ell = 2$ plays an important role to characterize the algebra $W_{q_1,q_2}(A_1)$. Namely, one can obtain the OPE of the generating current $\mathsf{T}_{1,x}$ [2]

$$f\left(\frac{x'}{x}\right) \mathsf{T}_{1,x} \mathsf{T}_{1,x'} - f\left(\frac{x}{x'}\right) \mathsf{T}_{1,x'} \mathsf{T}_{1,x} = -\frac{(1-q_1)(1-q_2)}{1-q} \left(\delta\left(q \frac{x'}{x}\right) - \delta\left(q \frac{x}{x'}\right) \right) \tag{37}$$

with the delta function $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$, and the scalar factor arising from the OPE of Y-operators $f(z) = \exp\left(\sum_{n=1}^\infty \frac{(1-q_1^n)(1-q_2^n)}{n(1+q^n)} z^n \right)$. See the commutation relation of the y-oscillators (30).

5. Affine quiver W-algebra

The formalism discussed in Sec. 4 is applicable to any quiver, including affine and hyperbolic quivers. Here let us consider the simplest affine quiver $\widehat{A}_0$, consisting of a single node with a loop edge. In this case, the mass deformation plays an essential role for the q-Cartan matrix:

$$c^{[m]} = 1 + q^{-m} - \mu^{-m} - \mu^m q^{-m} \overset{m \to 0}{\longrightarrow} 0 \tag{38}$$

The parameter $\mu \in \mathbb{C}^\times$ is called the multiplicative adjoint mass of 5d $\mathcal{N} = 1^*$ theory. In the massless limit, $\mu = 1$, the q-Cartan matrix becomes trivial.

The qq-character is generated by the local reflection

$$\mathsf{Y}_{1,x} \longrightarrow \mathcal{S}(\mu^{-1}) : \mathsf{Y}^{-1}_{1,q^{-1}} \mathsf{Y}_{1,\mu^{-1}x} \mathsf{Y}_{1,\mu q^{-1} x} : . \tag{39}$$

A remarkable feature of the affine quiver is that the qq-character does not terminate within finite terms since the affine character is in general given by an infinite series.

In addition, the coefficients appearing in the qq-character are given by the Nekrasov function with dual variables. See [5, 9] for details. In general, the qq-character of $\widehat{\Gamma}$-quiver theory on the ALE space $\widetilde{\mathbb{C}^2/\Gamma'}$ is dual to that of $\widehat{\Gamma}'$-quiver theory on $\widetilde{\mathbb{C}^2/\Gamma}$. This duality is interpreted as a generalization of [18], which is naturally understood using the 8-dimensional setup, called the gauge origami [16].

6. Quiver elliptic W-algebra

The quiver W-algebra discussed so far arises from the 5d quiver gauge theory defined on $\mathbb{R}^4 \times S^1$. Starting with the 6d gauge theory on $\mathbb{R}^4 \times T^2$, one can define the elliptic deformation of W-algebras. Let p be the elliptic nome $p = \exp(2\pi\sqrt{-1}\tau) \in \mathbb{C}^\times$ with the modulus of the torus T^2 denoted by τ. In this case the partition function is given by applying the elliptic class to the Chern characters,

$$\mathbb{I}_p\left[\sum_k x_k\right] = \prod_k \theta(x_k^{-1};p) \tag{40}$$

with $\theta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty$, which is reduced to the Dolbeault index in the limit $p \to 0$. To obtain the free field realization, we apply the Clavelli–Shapiro's doubling trick [19]. For example, the Y-operator is given by

$$\mathsf{Y}_{i,x} = \ q_1^{\tilde{\rho}_i} : \exp\left(y_{i,0} + \sum_{m\neq 0} \left(y_{i,m}^{(+)}\, x^{-m} + y_{i,m}^{(-)}\, x^{+m} \right) \right) : \tag{41}$$

with

$$\left[y_{i,m}^{(\pm)},\, y_{j,m'}^{(\pm)} \right] = \mp \frac{1}{m}(1-q_1^{\pm m})(1-q_2^{\pm m})\, \tilde{c}_{ij}^{[\mp m]}\, \delta_{m+m',0}\,. \tag{42}$$

See [6] for details.

7. Fractional quiver W-algebra

We define a fractional quiver (Γ, d), which is a quiver Γ decorated with a set of parameters $(d_i)_{i\in\Gamma_0}$. We assume $d_i \in \mathbb{Z}_{>0}$ so that it plays a role of the root length of the corresponding algebra. We define a gauge theory partition function depending on (d_i) by replacing the equivariant parameter as $(q_1, q_2) \to (q_1^{d_i}, q_2)$. The universal sheaf for the node $i \in \Gamma_0$ is given by

$$\mathbf{Y}_i = (1-q_1^{d_i}) \sum_{x\in\mathcal{X}_i} x =: \left(\sum_{r=0}^{d_i-1} q_1^r \right) \mathbf{y}_i \tag{43}$$

where $(\mathbf{y}_i)_{i\in\Gamma_0}$ is the fractionalization of the universal sheaf

$$\mathbf{y}_i = (1-q_1) \sum_{x\in\mathcal{X}_i} x\,. \tag{44}$$

The fractionalized sheaf plays a fundamental role to construct the fractional quiver W-algebra, which reproduces the q-deformed W-algebra of [8] for $\Gamma \neq ADE$. The symmetrization of the q-Cartan matrix (15) is then given by

$$b_{ij} = \frac{1-q_1^{d_i}}{1-q_1}(1+q_1^{-d_i}q_2^{-1})\delta_{ij} - \sum_{e:i\to j} \mu_e^{-1} \frac{(1-q_1^{d_i})(1-q_1^{-d_j})}{(1-q_1)(1-q_1^{-d_{ij}})} - \sum_{e:j\to i} \mu_e q_1^{-d_{ij}} q_2^{-1} \frac{(1-q_1^{d_i})(1-q_1^{-d_j})}{(1-q_1)(1-q_1^{-d_{ij}})} \tag{45}$$

where we omit the degree of the character and $d_{ij} = \gcd(d_i, d_j)$. See [7] for details.

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