An extended Steinberg group

A tool to detect non-singular closed 1-forms which are non-isotopic

Carlos MORAGA FERRÁNDIZ (The University of Tokyo)*

1. Motivation: The isotopy problem of non-singular closed 1-forms

The problem we are concerned with is the next one. Consider M^{n+1} a connected compact smooth manifold without boundary, of dimension $n+1$. Pick a de Rham cohomology class $0 \neq u \in H^1(M;\mathbb{R})$ and denote by Ω^u the topological space of closed 1-forms α in the class u with the $\mathcal{C}^\infty$-topology. We suppose that u contains *non-singular* representatives; in other terms, that the space

$$\Omega^u_{NS} := \{\alpha \in \Omega^u \mid \text{The set of zeroes } Z(\alpha) \text{ is empty}\}$$

is non-empty. Such an u can always be chosen if we suppose that M fibres over the circle, as it is shown in the brief paper [23]. We want to study Ω^u_{NS} up to isotopy.

Two closed 1-forms $\alpha_0, \alpha_1 \in \Omega^u$ are *isotopic* within the class u if there exists an isotopy $(\varphi_t)_{t\in[0,1]}$ of M preserving the class u such that $\varphi_1^*(\alpha_0) = \alpha_1$. If $\alpha_0, \alpha_1 \in \Omega^u$ are isotopic, they are clearly homotopic. The converse is also true if we restrict the attention to non-singular elements of Ω^u: we can integrate the homotopy to an isotopy by using a Moser type argument – see [20] – as it is done in [11, App. I]. We are so interested in $\pi_0(\Omega^u_{NS})$.
This π_0 is in general poorly understood. The more significant result is maybe that of [11]: Ω^u_{NS} is always connected in dimension 3. This is a very strong statement which has the difficult theorem of Cerf [4] about the nullity of Γ_4 as a corollary. However, Laudenbach proved in [12, Th.1] that $\pi_0(\Omega^u_{NS})$ is infinite in the case of rational cohomology classes on the torus $\mathbb{T}^m, m \geq 6$; also in those dimensions and for rational classes, [9] gave a collection of three obstructions to isotopy. Both papers deal only with rational classes u so that they can represent it by the homotopy class of a submersion $M \xrightarrow{p} \mathbb{S}^1$; the dimensional constraint arises because those papers make use of the crucial work of Hatcher and Wagoner [6] applied to the so called *pseudo-isotopy group* of $F := p^{-1}(\{*\})$. We sometimes refer to the work of [6] as the *exact context.*
The pseudo-isotopy group of a compact manifold F is in bijection with the set of connected components of $\mathcal{E}_F$, the subspace of $\mathcal{F}_F := \mathcal{C}^\infty(F\times[0,1],[0,1])$ consisting on functions with no critical points. Remark the analogy between the pairs $(\Omega^u, \Omega^u_{NS})$ and $(\mathcal{F}_F, \mathcal{E}_F)$. The set $\pi_0(\mathcal{E}_F)$ carries indeed a group structure; Hatcher and Wagoner proved that there exists an exact sequence of abelian groups

$$\mathrm{Wh}_1^+(\pi_1 F;\mathbb{Z}_2\times\pi_2 F) \xrightarrow{j} \pi_0(\mathcal{E}_F) \xrightarrow{\Sigma} \mathrm{Wh}_2(\pi_1 F) \longrightarrow 0 \ , \qquad (1)$$

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*e-mail: `carlos@ms.u-tokyo.ac.jp`

if $\dim(F)$ is 6 at least. The outward part of the sequence (1) are groups related to the K-theory of the group ring $\mathbb{Z}[\pi_1 F]$. This sequence is indeed presented as in [8, Th. 8.a.1], where an error on the initial proof – involved with the map j of (1) – was corrected. The result of Hatcher and Wagoner is remarkably profound: if we suppose F simply-connected, we retrieve the *pseudo-isotopy* theorem of Cerf [3].

Despite we cannot employ the theorem of [6] to our subject of study, we adapt their approach to define a map $\pi_0(\Omega^u_{NS}) \xrightarrow{\Sigma_u} \mathrm{Wh}_2(u)$ similar to the map Σ on (1); this is done on [16] for every pair (M,u) as before – u rational or not – when $n \geq 6$. We perform this under the same index assumptions that appear on theorem 3.1. As in the exact context, $\mathrm{Wh}_2(u)$ comes from a Steinberg-type group $\mathrm{St}(u)$ that we call the *u-extended Steinberg group* (see definition 3.1). The purpose of this note is to present it and to explain the geometric reasons that lead to $\mathrm{St}(u)$ by means of theorem 3.1.

1.1. Approach to define Σ_u

The more natural algebraic object we can associate with a closed 1-form α is the Morse-Novikov complex $\big(C_*(\alpha), \partial_\xi\big)$ which is defined when α is regular enough, namely of Morse type. The modules $C_*(\alpha)$ are freely generated by – a bijective lifting to the universal cover $\widetilde{M}$ of M of – the zeroes of α over the ring Λ_u, which is the Novikov completion of the group ring $\Lambda := \mathbb{Z}[\pi_1 M]$ by the morphism $\pi_1 M \to \mathbb{R}$ induced by $u \in H^1(M;\mathbb{R}) \approx \mathrm{Hom}(\pi_1 M, \mathbb{R})$; the map ∂_ξ depends on a contractible choice of an α-Lyapunov vector field ξ, that we call *equipment*. If ξ is regular enough, say Morse-Smale, the stable and unstable manifolds of ξ relative to the zeroes of ξ – which coincide with $Z(\alpha)$ – intersect transversally. We denote by $\mathscr{X}$ the whole set of α-equipments, and by $\mathscr{X}_{MS}$ the subset of Morse-Smale ones. By choosing orientations of the unstable manifolds, we obtain the map ∂_ξ which assigns coefficients in Λ_u to any pair of zeroes of consecutive index. This map results to be a differential, and the associated homology is independent of the choices we made. The unfamiliar reader can consult the source reference [21] or [5], [22] for further information on Morse-Novikov theory.

Fix $\alpha_0 \in \Omega^u_{NS}$ from now on. Since Ω^u is convex, so contractible, we have a bijection

$$\pi_1(\Omega^u, \Omega^u_{NS}; \alpha_0) \xrightarrow{\ \sim\ } \pi_0(\Omega^u_{NS})$$

which associates the connected component of α_1 with paths $(\alpha_t)_{t\in[0,1]}$ based on α_0. Morally, if $\pi_0(\Omega^u_{NS})$ was trivial, we could deform any path to another one where there is no significant modification along time. In order to follow the evolution of a path, we provide it with an equipment $(\xi_t)_{t\in[0,1]}$. Roughly speaking, Σ_u reads *bifurcations* that occur in $(\alpha_t,\xi_t)_{t\in[0,1]}$, consistently up to homotopy of the path; we are so led to study *generic* 2-parameter equipped families on Ω^u.

The generic property concerning $(\alpha_t)_{t\in[0,1]}$ is similar to that of a generic path of functions on $F\times[0,1]$: as in the exact context, the path is made up of Morse closed 1-forms except for a finite amount of *death/birth* times, where the 1-form presents a cubical-type zero and the Morse-Novikov complex *de/stabilises* with a pair of Morse zeroes of consecutive index.

The main difference with the exact context resides on the equipment part since the property for $(\xi_t)_{t\in[0,1]}$ of being Morse-Smale everywhere but in a finite amount of times – which holds on the exact context – is not generic at all for (α_t)-equipments: the set

$\Delta \subset [0,1]$ of bifurcation times where $(\xi_t)_{t\in[0,1]}$ is not Morse-Smale is infinite in general. The reason for that is the same that makes ∂_ξ to have coefficients on the "series-like" ring Λ_u rather than on the "polynomial-like" ring Λ: the orbits of an α-equipment ξ tend to wrap around themselves. In order to elude this problem, we introduced the L-transversality condition ($L > 0$) in [17, §2.1.5]. This condition is a *truncated* version of the Morse-Smale one:
By fixing a base point on M and a path to each zero of α, we can determine an element $g \in \pi_1 M$ each time we find a ξ-orbit between zeroes of α. The L-transversality condition asks that every ξ-orbit inducing such a $g \in \pi_1 M$ and verifying $u(g) > -L$, comes from a transversal intersection of the un/stable manifolds concerned with the orbit. We denote the set of L-transverse equipments by $\mathscr{X}_0^L$. Clearly we have $\cap_{L>0}\mathscr{X}_0^L = \mathscr{X}_{MS}$. An L-incidence matrix can be still defined for these vector fields: their coefficients belong to the L-truncation of the Novikov ring $\Lambda_u^L := \mathrm{tr}_L(\Lambda_u)$ (see subsection 2.1). An equipment $(\xi_t)_{t\in[0,1]}$ is generically L-transversal everywhere but in a finite amount of *bifurcation* times, where we say that the equipment is *L-handle-slide*; these vector fields verify the condition of $\mathscr{X}_0^L$ except for a single orbit – between zeroes of same index – whose coefficient also verifies $u(g) > -L$. We denote them by $\mathscr{X}_1^L$. The generic (α_t)-equipments just described are so paths in $\mathscr{X}_0^L \cup \mathscr{X}_1^L$, and are called L-generic.
Bifurcation times are called L-handle-slide because they have an homological effect similar to that of the operation described on [14, Th. 7.6]: if we denote by $A_*^\pm$ the L-incidence matrices respectively before/after crossing such an accident concerning a couple of points of index i, there exists an *u-extended elementary matrix* E – as in subsection 2.2 – involved with the coefficient $g \in \pi_1 M$ associated with the orbit such that:

$$\left\{\begin{array}{rcll} A_j^+ &=& A_j^- &, \quad \text{if } j \neq i, i+1 \\ A_i^+ &=& EA_i^- & \\ A_{i+1}^+ &=& A_{i+1}^- E^{-1} & \end{array}\right. \tag{2}$$

up to L-truncation, as it is shown on [17, Prop. 2.2.36]. The map Σ_u counts handle-slide bifurcations in a convenient way.

A second complication not happening on the exact context, is a special type of handle-slide that we call *self-sliding*: an orbit from a Morse zero to itself appears. These bifurcations accidents were mentioned in Latour's paper [10], where he found an algebraic characterisation of classes $0 \neq u \in H^1(M;\mathbb{R})$ such that $\Omega_{NS}^u \neq \varnothing$. We interpret his theorem as a sort of s-cobordism theorem (consult the short note [19]), where the vanishing of a torsion $\tau(M,u)$ appears as an obstruction for Ω_{NS}^u being non-empty. This torsion lives in a Whitehead-type group: employing the notations of our section 2, this group is $\mathrm{Wh}_1(u) := \frac{K_1(\Lambda_u)}{\langle \pm\pi_1 M,\ 1+(u<0)\rangle}$. Unfortunately, Latour omitted the analysis of self-slidings: "*... This replaces a long study of* ***homoclinic bifurcations*** *in an earlier version which had the advantage of indicating the geometric reason to divide $K_1(\Lambda_u)$ by trivial units...*" [1]. In fact, the equalities (2) explain Latour's words: in the case of self-slidings, the matrix E is an u-elementary matrix as in our definition 2.1. These matrices are elements of $\mathrm{GL}(\Lambda_u) \setminus \mathrm{E}(\Lambda_u)$ and survive on $K_1(\Lambda_u) := \frac{\mathrm{GL}(\Lambda_u)}{\mathrm{E}(\Lambda_u)}$. To calculate the torsion, one needs to choose a Morse-Smale equipment ξ. Near a self-sliding accident, we can find two different such choices ξ_0, ξ_1 such that the related torsions would differ by $\tau(E)$, and would not coincide on $\frac{K_1(\Lambda_u)}{\langle \pm\pi_1 M\rangle}$. Latour needed hence

[1] This is a translation of a comment on the third page of [10].

to mod out by the trivial units $1+\big(u<0\big)$. These accidents deeply enrich the theory and play a fundamental role on the geometry of the extended Steinberg group.

2. The algebraic framework of the (non-exact) isotopy problem

We recall the definition of the *standard* Steinberg group $\mathrm{St}(R)$ associated with an *associative and unitary* ring R. For more details, consult [13, §5].

For n any positive integer, let $\mathrm{GL}_n(R)$ denote the set of $n\times n$ invertible matrices with coefficients on R. We denote the direct limit induced by the natural sequence of inclusions $\mathrm{GL}_n(R)\hookrightarrow \mathrm{GL}_{n+1}(R)$ by $\mathrm{GL}(R)$. The subgroup of *elementary matrices* $\mathrm{E}(R)$ is generated by the set $\big\{e_{ij}^r \;\big|\; i\neq j\in\mathbb{N}^*, r\in R\big\}$ where $e_{ij}^r=\mathrm{Id}+t_{ij}^r$ and t_{ij}^r denotes the matrix whose only non-necessarily zero term is r on the (i,j) component. We define the Steinberg group associated with R by presentation. The generators are given by the set $\big\{x_{ij}^r \;\big|\; i\neq j\in\mathbb{N}^*, r\in R\big\}$ and the relations are the so-called Steinberg relations:

$$\left\{\begin{array}{lcll}(\mathrm{RS}_1) & \equiv & x_{ij}^r x_{ij}^s = x_{ij}^{r+s} & \\ (\mathrm{RS}_2) & \equiv & [x_{ij}^r, x_{kl}^s]=1 & \text{, if } i\neq l, j\neq k \\ (\mathrm{RS}_3) & \equiv & [x_{ij}^r, x_{jl}^s]=x_{il}^{rs} & \text{, if } i\neq l.\end{array}\right. \tag{3}$$

Sometimes they are also called the *trivial relations* of elementary matrices: one can easily verify that they hold true if we replace the symbol $x_{\cdot\cdot}^{\bullet}$ by $e_{\cdot\cdot}^{\bullet}$. Thus the map $\varphi:\mathrm{St}(R)\to\mathrm{E}(R)$ given by $x_{ij}^r\mapsto e_{ij}^r$ is a surjective group morphism and the Steinberg group becomes a relevant actor on low algebraic K-theory due to the next exact sequence:

$$0\longrightarrow K_2(R)\longrightarrow \mathrm{St}(R)\xrightarrow{\varphi}\mathrm{GL}(R)\longrightarrow K_1(R)\longrightarrow 0 \tag{4}$$

where the second and first K-groups of R are seen respectively as the kernel and cokernel of φ. The group $\mathrm{Wh}_2(\pi_1F)$ appearing in (1) is in fact a quotient of $K_2(R)$ for $R=\mathbb{Z}[\pi_1F]$; at each bifurcation time $t=t_0$ of a generic equipment $(\xi_t)_{t\in[0,1]}$ of a generic path of functions $(f_t:F\times[0,1]\to[0,1])_{t\in[0,1]}$, we find an orbit of ξ_{t_0} from p_i to p_j, two Morse critical points of f_{t_0} of *same index*; a signed element $\pm g\in\pi_1F$ – the bifurcation coefficient[2] – can be associated with this orbit in a similar way to that one explained in subsection 1.1 and hence a generator $x_{ij}^{\pm g}$ of $\mathrm{St}\big(\mathbb{Z}[\pi_1F]\big)$; the magic fact is that, in the absence of birth/death singularities (functions only having $r\geq 3$ critical points of same index), the relative homotopy of generic paths $(f_t,\xi_t)_{t\in[0,1]}$ is governed by the Steinberg group. More precisely, if $\mathfrak{a}$ represents the mentioned set of functions f provided with an equipment ξ, and $\mathfrak{b}$ denotes the subset of $\mathfrak{a}$ such that ξ is Morse-Smale, for every fixed $(f_0,\xi_0)\in\mathfrak{b}$ we have:

$$\pi_1\big(\mathfrak{a},\mathfrak{b};(f_0,\xi_0)\big)\approx \mathrm{St}(r,\mathbb{Z}[\pi_1F]). \tag{5}$$

The reader can find the isomorphism (5) in [6, Ch.II, §1] and is aimed to compare it to theorem 3.1 of this document.

In simpler terms, when a deformation of $(f_t,\xi_t)_{t\in[0,1]}$ permutes the order in time of bifurcation accidents, we see the Steinberg relations appear. The map Σ on the sequence (1) is just a wise combination of the $x_{ij}^{\pm g}$ associated with bifurcations in order to have a well-defined map up to a homotopy of $(f_t,\xi_t)_{t\in[0,1]}$. But our work is naturally involved with Novikov rings…

[2] As a remark and by following [15, §9], a self-indexing Morse function f on $F\times[0,1]$ allows to determine the torsion of the cobordism $F\times[0,1]$ – thus trivial in this case – by means of a free $\mathbb{Z}[\pi_1F]$-complex $\big(\overline{C}_*,\partial_f\big)$ induced by f. The modification suffered by this complex when crossing a bifurcation time is exactly described by equations (2), where $E=\varphi(x_{ij}^{\pm g})$.

2.1. Searching an u-extension of the Steinberg group

We will denote π the fundamental group of our manifold M, as well as $\Lambda := \mathbb{Z}[\pi]$ for the sequel. A series $\lambda \in \mathbb{Z}^\pi$ is written $\sum_{g\in\pi} \lambda_g g$ where $\lambda_g = \lambda(g) \in \mathbb{Z}$. The *support* of such an element is given by $\mathrm{supp}(\lambda) := \{g \in \pi \mid \lambda_g \neq 0\}$. Recall that the Novikov ring associated with u is given by:

$$\Lambda_u := \left\{\lambda \in \mathbb{Z}^\pi \;\middle|\; \mathrm{supp}(\lambda) \cap u^{-1}\big([L, +\infty)\big) \text{ is finite for every } L \in \mathbb{R}\right\}.$$

Each $L \in \mathbb{R}$ defines a *truncation map*, $\mathrm{tr}_L : \Lambda_u \to \Lambda_u$ given by $\lambda \mapsto \sum_{g\in u^{-1}([L,\infty))} \lambda_g g$, which clearly factors through the inclusion $\Lambda \hookrightarrow \Lambda_u$. Denote $\Lambda_u^L := \mathrm{Im}\,(\mathrm{tr}_L)$ the L-truncation of the Novikov ring. Indeed, Λ_u^L and the quotient $\frac{\Lambda_u}{(u<L)}$ are isomorphic as abelian groups[3]. However, Λ_u^L does not inherit the ring product structure from Λ_u: the set $\big(u < L\big)$ is clearly not ideal of Λ_u. In other terms, the map tr_L is not a ring morphism, as we easily see by taking any $L > 0$ and $g \in \pi$ such that $-L < u(g) < -\frac{L}{2}$; clearly $\mathrm{tr}_{-L}(g^2) = 0 \neq g^2 = \mathrm{tr}_{-L}(g)\,\mathrm{tr}_{-L}(g)$. We can still define a product operation $\overset{*}{L}$ on Λ_u^L by setting $\lambda \overset{*}{L} \mu := \mathrm{tr}_L(\lambda\mu)$.

Knowing Hatcher and Wagoner's theory, and having L-transversality at hand, one is tempted to define a map Σ_u^L by employing $\mathrm{St}(\Lambda_u^L)$, the L-truncated version of the Steinberg group, and verbatim copying the definition of the map Σ on sequence (1); then trying to prove that this hypothetical Σ_u^L results on a well defined map, up to homotopy of $(\alpha_t, \xi_t)_{t\in[0,1]}$. This approach results to be catastrophic for many reasons:

- The main headache of this tentative is that the L-truncation ring $\big(\Lambda_u^L, +, \overset{*}{L}\big)$ has a bad behaviour due to the fact that tr_L is not a ring morphism: suppose $\pi \approx \mathbb{Z}$ generated multiplicatively by $\langle t\rangle$; take the morphism u given by $u(t) = -2$ and $L = -3$. Remark that $\mathrm{tr}_L(t^2) = 0$. In this example, Λ_u are Laurent series on t with bounded negative exponents and Λ_u^L are Laurent polynomials on t having exponents lower or equal to 1. Consider the products:

$$\begin{cases} (1+t) \overset{*}{L} \big(t \overset{*}{L} t^{-1}\big) & = & (1+t) \overset{*}{L} 1 = 1 + t \\ \big((1+t) \overset{*}{L} t\big) \overset{*}{L} t^{-1} & = & t \overset{*}{L} t^{-1} = 1 \end{cases}.$$

  We observe that the ring Λ_u^L is *not associative*[4] in general! And $\mathrm{St}(\Lambda_u^L)$ is even not defined since the Steinberg group makes sense only for unitary associative rings.

- We encounter an even deeper problem if we pretend to mimic the strategy of [6] to define Σ_u. We need to associate a symbol with each bifurcation, say of an L-generic equipped path $(\alpha_t, \xi_t)_{t\in[0,1]}$. Keeping the notations of the explanation just before the beginning of this subsection, the subscript part of the Steinberg element in the case of functions depended on the numbering of the critical points of f_{t_0}. The subscripts related with the bifurcation were always different because the f-Lyapunov condition implies $f_{t_0}(p_i) > f_{t_0}(p_j)$; returning to the context of closed 1-forms, the α_{t_0}-Lyapunov condition implies that ξ_{t_0}-orbits are transverse to the foliation induced by α_{t_0}; typically, these orbits will revisit the leaves of the foliation. A generic one-parameter family $(\xi_t)_{t\in[0,1]}$ will thus contain *self-sliding* bifurcations, and the subscripts of an hypothetical Steinberg symbol $x_{ij}^{\pm g}$ representing it should verify $i = j$. There is so no reasonable symbol to represent self-slindings on *any* standard Steinberg group!

[3] The notation $\big(u < L\big)$ refers to the subgroup of elements whose support is included in $u^{-1}(-\infty, L)$.
[4] Even worse, also non-unitary if $L > 0$ because $u(1_\pi) = 0$.

Remember that we are trying to distinguish equipped paths $(\alpha_t, \xi_t)_{t\in[0,1]}$ up to homotopy. A new dilemma arises now: there is no *a priori* reason to think that self-slidings should vanish up to homotopy; in other words, there is no *a priori* reason to positively answer the question:

Can we deform a generic $(\alpha_t, \xi_t)_{t\in[0,1]}$, fixing its extremities, into a generic $(\alpha'_t, \xi'_t)_{t\in[0,1]}$ containing no self-sliding?

The author has been working on giving a positive answer when the generic condition is L-genericness. The best we obtain is that self-slidings of $(\alpha_t, \xi_t)_{t\in[0,1]}$ can be replaced by self-slidings whose bifurcation coefficients have a lower u-value[5], until we obtain a path with no L-self-sliding. Even after this effort, a map Σ_u well-defined up to homotopy, should be invariant up to raising the value of L as much as we want; however, by choosing higher values L' we would have to push again L'-self-slidings, and this procedure may not end into a trivial operation up to homotopy. The more reasonable attitude to take is trying to construct an algebraic model bearing the existence of self-slidings: this is the aim of the *u-extended Steinberg group*.

2.2. The group of u-extended elementary matrices.

The subset $\big(u<0\big) \subset \Lambda_u$ is multiplicative. For any $\lambda \in \big(u<0\big)$, the series of powers $\lambda^+ := \sum_{i=1}^{\infty} \lambda^i$ belongs therefore to Λ_u and $1+\lambda^+$ turns to be the inverse of $1-\lambda$. For every such a λ, we denote $\lambda^- := -\lambda$ so that $1+\lambda^+$ and $1+\lambda^-$ are mutually inverse. For every $i \in \mathbb{N}^*$, denote by $t_{ii}^{\lambda^\pm}$ the matrix having $\lambda^\pm$ on the (i,i) entry as only non-zero term. As $\Lambda_u^\times = \mathrm{GL}_1(\Lambda_u)$, the mutually inverse matrices $e_{ii}^{\lambda^\pm} := \mathrm{Id} + t_{ii}^{\lambda^\pm}$ belong to $\mathrm{GL}(\Lambda_u)$; we call them *u-elementary* matrices. The matrix e_{ii}^{λ} denotes either $e_{ii}^{\lambda^+}$ or $e_{ii}^{\lambda^-}$.

Definition 2.1. We denote by $\mathrm{E}(u)$ the subgroup of $\mathrm{GL}(\Lambda_u)$ generated by the matrices

$$\left\{ e_{ii}^{\lambda}, e_{ij}^{\theta} \;\middle|\; \begin{array}{l} i,j \in \mathbb{N}^*,\;\; i \neq j \\ \lambda \in \big(u<0\big),\;\; \theta \in \Lambda_u \end{array} \right\}. \tag{6}$$

We call $\mathrm{E}(u)$ the group of *u-extended* elementary matrices.

Clearly, usual elementary matrices $\mathrm{E}(\Lambda_u)$ form a subgroup of $\mathrm{E}(u)$.

Lemma 2.1. *The following relations are verified on* $\mathrm{E}(u)$*:*

$$\left\{ \begin{array}{lll} (\mathrm{RS}_i)_{i=1,2,3} & \equiv & \text{As in (3)} \\ (\mathrm{RS}_1^u) & \equiv & e_{ii}^{\lambda} e_{ii}^{\mu} = e_{ii}^{\lambda+\mu+\lambda\mu} \\ (\mathrm{RS}_{2,a}^u) & \equiv & [e_{ii}^{\lambda}, e_{jj}^{\mu}] = 1 \qquad\qquad\qquad\;\; , \textit{if } i \neq j \\ (\mathrm{RS}_{2,b}^u) & \equiv & [e_{ii}^{\lambda}, e_{jk}^{\theta}] = 1 \qquad\qquad\qquad\;\; , \textit{if } i \neq j \neq k \neq i \\ (\mathrm{RS}_{3,a}^u) & \equiv & [e_{ii}^{\lambda^\pm}, e_{ij}^{\theta}] = e_{ij}^{\lambda^\pm \theta} \qquad\qquad\quad , \textit{if } i \neq j \\ (\mathrm{RS}_{3,b}^u) & \equiv & [e_{ii}^{\lambda^\pm}, e_{ji}^{\theta}] = e_{ji}^{\theta\lambda^\mp} \qquad\qquad\quad , \textit{if } i \neq j \\ (\mathrm{RS}_{4,a}^u) & \equiv & e_{ii}^{(\lambda\mu)^+} \big(e_{ij}^{-\lambda} e_{ji}^{\mu}\big) e_{jj}^{(\mu\lambda)^-} = e_{ji}^{\mu} e_{ij}^{-\lambda} \;\; , \textit{if } i \neq j \;\text{ and }\; \lambda\mu \in \big(u<0\big) \\ (\mathrm{RS}_{4,b}^u) & \equiv & e_{jj}^{(\mu\lambda)^-} \big(e_{ij}^{-\lambda} e_{ji}^{\mu}\big) e_{ii}^{(\lambda\mu)^+} = e_{ji}^{\mu} e_{ij}^{-\lambda} \;\; , \textit{if } i \neq j \;\text{ and }\; \lambda\mu \in \big(u<0\big). \end{array} \right.$$

[5] This self-sliding replacement is somewhat elaborate geometrically: a self-sliding with bifurcation coefficient g entails the gain/loss of a periodic *closed orbit* on the conjugacy class of g as [7, Rem. 3.12] mentioned (consult [1, §5.6.12 and Fig. 5.6-7]). The extremities of two self-sliding paths starting from (α_0, ξ_0) and creating closed orbits of respective class g and g^2, can be joint by a generic path of equipments containing an Andronov-Hopf (*period-doubling*) bifurcation, where the orbit related to g doubles its period to become the orbit related to g^2; this bifurcation is outstandingly well explained on [2, §34.C and Fig. 141].

Proof. We concentrate on the new $(\mathrm{RS}^u_\bullet)$-relations concerning u-elementary matrices. The first one is trivial, as well as the relations of second type $(2 \in \bullet)$ since non-trivial coefficients are on the diagonal and they do not interact when computing the products thanks to $i \neq j$. We can safely suppose than $i = 1, j = 2$ from now on; relation $(\mathrm{RS}^u_{3,b})$ is an straightforward calculation, but a slight subtlety is needed for $(\mathrm{RS}^u_{3,a})$: we have $[e_{ii}^{\lambda^\pm}, e_{ij}^\theta] = \left(\begin{smallmatrix} 1+\lambda^\pm & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \theta \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+\lambda^\mp & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\theta \\ & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1+\lambda^\pm & \theta+\lambda^\pm\theta \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+\lambda^\mp & -\theta-\lambda^\mp\theta \\ & 1 \end{smallmatrix}\right)$. After simplification, this product equals $e_{ij}^{-(\lambda^\mp+\lambda^\pm\lambda^\mp)\theta}$.
Notice that λ^+ and λ^- commute; if we consider their product:

$$\lambda^+\lambda^- = \lambda^-\lambda^+ = -\sum_{i\geq 2} \lambda^i = -\left(\lambda^+ + \lambda^-\right), \tag{7}$$

we conclude that the superscript on the elementary matrix we just found equals $\lambda^\pm\theta$. The fourth-type relations are of new nature since there were no trivial relation over standard elementary matrices concerning the elementary generators $e_{ij}^\bullet, e_{ji}^\bullet$. These new relations say that the just mentioned standard elementary matrices of the Novikov ring commute up to u-elementary matrices. The $(4,b)$-relation can be deduced from $(4,a)$ just by moving the u-elementary terms to the other side of the equality and changing the roles of the actors on the pairs (i,j) and $(-\lambda,\mu)$. Focusing on relation $(4,a)$, let us call X, Y the products inside the parentheses and on the right-side of the equality respectively. After calculation, we find the matrices $X = \left(\begin{smallmatrix} 1-\lambda\mu & -\lambda \\ \mu & 1 \end{smallmatrix}\right), Y = \left(\begin{smallmatrix} 1 & -\lambda \\ \mu & 1-\mu\lambda \end{smallmatrix}\right)$. Since $\lambda\mu \in \big(u < 0\big)$, the same happens to $\mu\lambda$ and the terms different from 1 on the diagonals of X, Y are invertible. We easily see that multiplying X with the mentioned u-elementary matrices will provide a matrix with the diagonal terms of Y. The element μ on the (j,i) entry remains unchanged in doing so, and the (i,j) term becomes $-\big(1+(\lambda\mu)^+\big)(\lambda - \lambda\mu\lambda)$. Remark now that:

$$(\lambda\mu)^+\lambda = \lambda\mu\lambda + \left(\sum_{i\geq 2}(\lambda\mu)^i\right)\lambda = \lambda\mu\lambda + \left(\sum_{i\geq 2}(\lambda\mu)^{i-1}\right)\lambda\mu\lambda = \lambda\mu\lambda + (\lambda\mu)^+\lambda\mu\lambda. \tag{8}$$

Using (8) while expanding the product we just found, one easily ends up with $-\lambda$. $\square$

The geometric *raison d'être* of $\mathrm{E}(u)$ is, as we mentioned before, equations (2) describing how the map ∂_t counting flow lines of ξ_t on an L-generic equipment changes when crossing the finite set of time bifurcations. Since we are interested on paths $(\alpha_t, \xi_t)_{t\in[0,1]}$ up to homotopy – say depending on $s \in [0,1]$ – bifurcation times may vary their order of appearance on t for different fixed values of s; in other terms, for some isolated values of (t,s) two orbits of L-handle-slide type appear. This situation is unavoidable on 2-parameter families of equipments. We call *L-crossing* the equipments concerning these isolated values and denote them by $\mathscr{X}^L_{2,c}$. At L-crossing parameters, an interaction between the involved bifurcations can take place. These interactions are precisely described by the relations of the u-extended Steinberg group of definition 3.1. In the converse terms, the group $\mathrm{St}(u)$ is geometrically realized by the relative homotopy classes of – some – generic paths $(\alpha_t, \xi_t)_{t\in[0,1]}$: this is the content of theorem 3.1. The condition about λ on generators x_{ii}^λ of $\mathrm{St}(u)$ is explained by the fact that any generic self-sliding bifurcation, comes with a *bifurcation coefficient* $g \in \pi, u(g) < 0$, a *sign* $\pm$ and a *dichotomic character* $(\cdot)^\pm$. Once the coefficient g has been determined, there exist – up to sign – two geometrically non-equivalent generic bifurcation behaviours: if we study the accident with dichotomic character $(\cdot)^+$ on the universal cover $\widetilde{M}$, the traces after the bifurcation of a lifting of the unstable manifold concerned with the accident on subsequently lower levels of $\widetilde{M}$, are an *iterated connected sum* of the $(g^i, i \geq 0)$-translated copies of the trace before the accident, as figure 1 suggests[6]. The mentioned unstable manifold is $W^u(p^k; \xi_t)$. This explains the presence of terms $(\pm g)^{\pm}$ on the definition of the u-extended Steinberg group $\mathrm{St}(u)$.

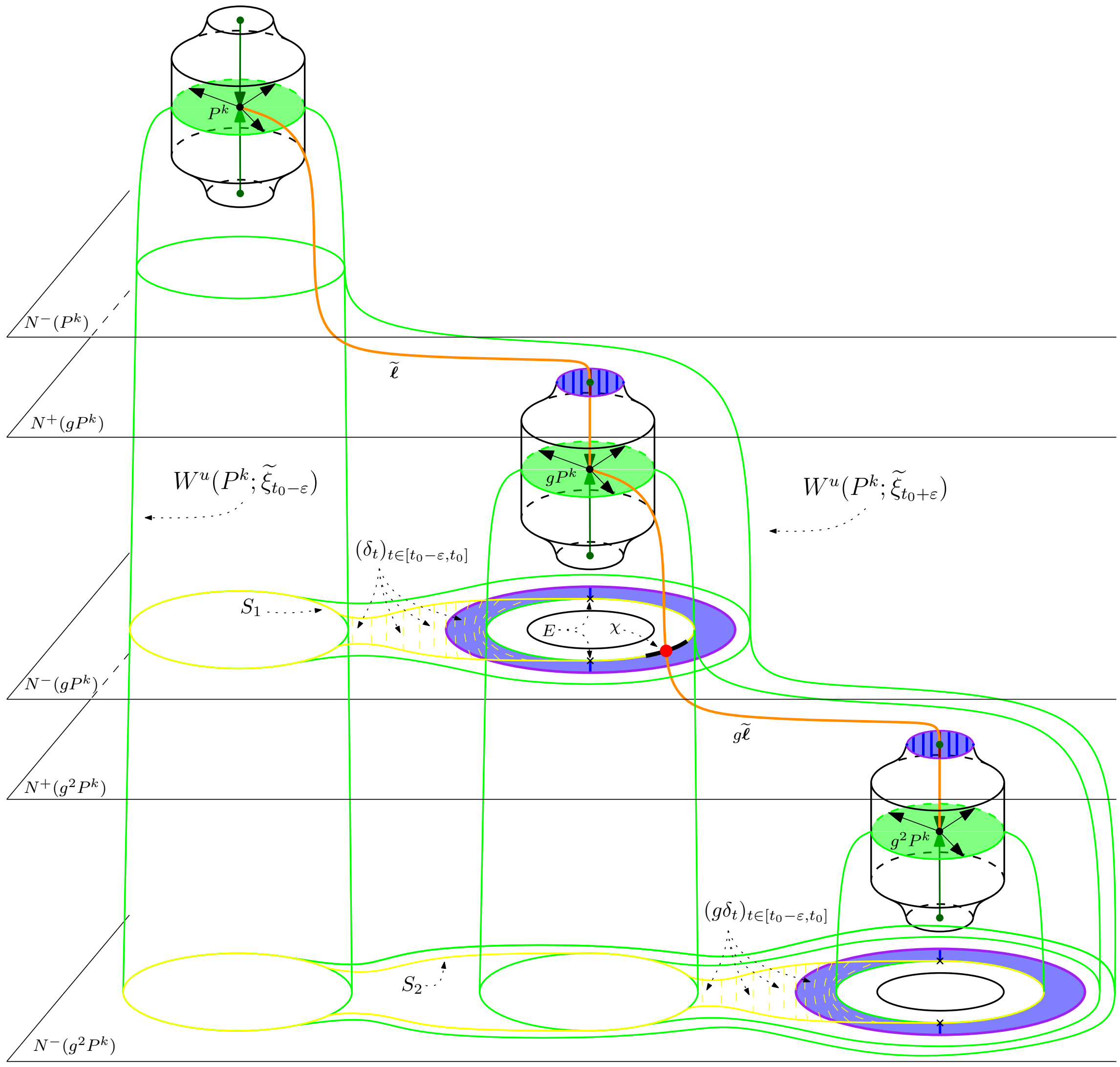

Figure 1: Situation before/after a $(g)^+$-self-sliding of orbit $\boldsymbol{\ell}$ and dichotomy point χ.

Moreover, when an unstable manifold slides over itself twice simultaneously at $t = t_0$, say with coefficients $g, h \in \pi$ where $u(g), u(h) < 0$, we cannot circumvent the appearance of a *resonance* phenomenon – of coefficient gh – at the same time $t = t_0$. This resonance factor is detected by relation (RS_1^u) of lemma 2.1.

3. The u-extended Steinberg group $\mathrm{St}(u)$ and its geometry

Definition 3.1. Let $r \geq 3, r \in \mathbb{N} \cup \{\infty\}$. If $(\cdot)^{\pm}$ stand for the operators described at the beginning of subsection 2.2, we define the u-extended Steinberg group of order r as the group having a generator-relation presentation where generators are given by the

[6] The interested reader may consult [17, p.51, 2$^{\text{nd}}$ case: $k = l$].

set:

$$\left\{ x_{ii}^{\lambda}, x_{ii}^{\lambda+\mu+\lambda\mu}, x_{ij}^{\theta} \;\middle|\; \begin{array}{l} i,j \in \{1,\dots,r\},\ i \neq j \\ \lambda,\mu \in \left\{ (\pm g)^{\pm} \;\middle|\; g \in \pi, u(g) < 0 \right\} \cup \{0\},\ \theta \in \Lambda_u \end{array} \right\},$$

and relations are as those appearing in lemma 2.1, after replacing "e" with "x". We denote this group by $\mathrm{St}(r,u)$.

Remark 3.1. As it happened in the usual Steinberg group, the elements x_{ii}^0 represent the identity element because $x_{ii}^0 x_{ii}^0 = x_{ii}^0$ is verified thanks to relation (RS_1^u). The same relation tells us that $x_{ii}^{\lambda^+}$ and $x_{ii}^{\lambda^-}$ are mutually inverse because $x_{ii}^{\lambda^+} x_{ii}^{\lambda^-} = x_{ii}^{\lambda^+ + \lambda^- + \lambda^+\lambda^-} = x_{ii}^0$, the last equality coming from (7).

We introduce and motivate the remainder necessary notions to state theorem 3.1.

Definition 3.2. For $\alpha \in \Omega^u$ of Morse type, denote $r_i(\alpha)$ the cardinal of $Z_i(\alpha)$, its set of zeroes of index i. Any $(r_0,\dots,r_{n+1}) \in \mathbb{N}^{n+2}$ such that $\sum_{i=0}^{n+1}(-1)^i r_i = 0$ is called *admissible*. For admissible $(n+2)$-tuples we denote:

$$\Omega^u_{(r_0,\dots,r_{n+1})} := \{\alpha \in \Omega^u \mid r_i(\alpha) = r_i\}.$$

In addition, for every $\mathscr{Y} \subset \mathscr{X}$, we denote by $\Omega^{u,\mathscr{Y}}_{(r_0,\dots,r_{n+1})}$ the space of pairs (α,ξ) such that $\xi \in \mathscr{Y}$ is an α-equipment, with $\alpha \in \Omega^u_{(r_0,\dots,r_{n+1})}$.

Remark 3.2. Of course the zero element of $\mathbb{N}^{n+2}$ gives $\Omega^u_{(0,\dots,0)} = \Omega^u_{NS}$. Any non-admissible $(n+2)$-tuple of non-negative numbers provides the empty set: since Ω^u_{NS} is non-empty, we know that the Novikov complex is acyclic. In particular, its Euler characteristic must be zero. This is exactly the condition for admissible tuples.

We explain now the index condition (9) of theorem 3.1. The bijection of the theorem should reflect the simplest relation (RS_1) in the geometrical side. One finds homotopical obstructions to that relation if there are zeroes p_i, p_j having index or coindex lower or equal than 2. Suppose that a path $(\alpha_t,\xi_t)_{t\in[0,1]}$ has two consecutive accidents corresponding to x_{ij}^g, x_{ij}^{-g}; one can construct a loop γ inside a level F of $\widetilde{M}$, that is nulhomotopic by a disk $\mathbb{D}^2 \subset \widetilde{M}$. If we are able to **push** $\mathbb{D}^2$ into an **embedded** disk on the level F, we can then construct a Whitney isotopy of $(\alpha_t)_{t\in[0,1]}$ leading to another generic path which does not contain any more the mentioned accidents: we have unknotted the product $x_{ij}^g x_{ij}^{-g}$ to x_{ij}^0. Here, the index and dimension conditions appear: in order to push $\mathbb{D}^2$ to F without introducing new accidents, we should continuously deform $\mathbb{D}^2$ by following the flow lines of ξ_t. If there exists a $q \in Z_i(\alpha)$ contradicting the second condition appearing in (9), either the stable or the unstable manifold of q intersects $\mathbb{D}^2$ by a basic general position argument[7] and we cannot push our disk into F. We further need $\mathbb{D}^2$ to be embedded in F to construct the Whitney isotopy; this is not true in general, but we can suppose it for granted if $5 \le \dim(F) = \dim(M) - 1$ thanks to the Whitney embedding theorem. Compare to [6, Ch. II, §1, Lemma 1.2'(a)].

Remark 3.3. The second condition on (9) requires the dimension of M to be at least 5. But at this dimension there is no r_i different from zero and there is nothing to prove. Dimension 6 only allows r_3 to be non-zero, which is impossible for admissible tuples of this length. We hence require $\dim(M) \ge 7$.

[7] This can be summarised by saying that $i_* : \pi_1(X \setminus A) \to \pi_1(X)$ is an isomorphism if $\mathrm{cod}_X(A) \ge 3$ when A and X are smooth.

There is another unavoidable accident on 2-parameter families of equipments, called *L-exchange*, that we had not yet mentioned: ξ_t^s has, for isolated values of (t,s), a single orbit from a zero of index $i-1$ to another one of index i, whose coefficient verifies $u(g) > -L$. We designate them by $\mathscr{X}_{2,e}^L$.

Definition 3.3. A 2-parameter family (ξ_t^s) of equipments is said to be L-generic if there exists a finite set $\Delta_2 \subset (0,1)$ such that for all $s \notin \Delta_2$, the path $(\xi_t^s)_{t\in[0,1]}$ is an L-generic path and for every $s \in \Delta_2$, there exists an unique $t_s \in (0,1)$ such that $\xi_{t_s}^s \in \mathscr{X}_{2,c}^L \cup \mathscr{X}_{2,e}^L$.

Remark 3.4. Truncations are useful to realise the analysis and to construct parameter families (α_t, ξ_t); but this does not mean that an isotopy obstruction should have a "truncated type"; even in the situation without parameters of Latour's paper, the torsion obstruction $\tau(M,u)$ naturally lives in a quotient of $K_1(\Lambda_u)$ and not in a algebraic object based on the truncations Λ_u^L of the Novikov ring.
Another clue telling us that a truncated isotopy obstruction is not plausible, is the next simple argument: after inspection, one realizes that the homotopy relations of the exact context (the usual Steinberg relations) are still verified when considering homotopies of paths $(\alpha_t, \xi_t)_{t\in[0,1]}$. In particular, the relation (RS_3) concerning the accidents x_{ij}^g, x_{jl}^h where $u(g), u(h) \in (-L, -\frac{L}{2})$ holds. But Σ_u^L cannot detect the resulting interaction x_{il}^{gh} because $u(gh) < -L$.

Theorem 3.1. *Let $(r_0, \ldots, r_{n+1})$ be admissible as in definition 3.2. Suppose that:*

$$\begin{cases} r_i \geq 3 \quad \text{or} \quad r_i = 0 & \text{if } i \in \{3, \ldots, n-2\} \\ r_i = 0 & \text{otherwise.} \end{cases} \tag{9}$$

Fix $(\alpha_0, \xi_0) \in \Omega_{(r_0,\ldots,r_{n+1})}^{u,\mathscr{X}_{MS}}$. There exists a bijection:

$$\chi^u : \pi_1\left(\Omega_{(r_0,\ldots,r_{n+1})}^{u,\mathscr{X}}, \Omega_{(r_0,\ldots,r_{n+1})}^{u,\mathscr{X}_{MS}}; (\alpha_0,\xi_0)\right) \xrightarrow{\ \simeq\ } \bigoplus_{r_i \neq 0} \mathrm{St}(r_i, u).$$

Idea of proof. We briefly explain the proof that will appear on [16]. The property of definition 3.3 is proved to be generic and open for 2-parameter equipments, and this for every $L > 0$. The intersection for every $L \in \mathbb{N}^*$ of L-generic 2-parameter families, that we denote by $\mathscr{X}_{0,1,2}$ is thus a residual set in the Baire space $\mathscr{X}$; we can thus approach the equipment of any homotopy class by a family on $(\mathscr{X}_{0,1,2}, \mathscr{X}_{MS})$, where we understand the occurring bifurcations. The map χ^u collects the L-handle-sliding bifurcations for increasing L. Accumulation of bifurcations do not create a problem because we can rearrange bifurcations in time in such a way that accidents concerning different subscripts (i,j) are not mutually mixed in time! This was not possible in the context case, but here, relations (RS_4^u) allow one to do so. This ends with a well-defined element of $\mathrm{St}(r_i,u)$ for each critical index i. $\square$

In order to define the map $\Sigma_u : \pi_0(\Omega_{NS}^u) \to \mathrm{Wh}_2(u)$ that we mentioned on section 1, we need theorem 3.1. The index hypothesis is indeed not so restrictive: the first condition can be achieved by introducing as many trivial pairs of zeroes of consecutive indexes as needed. For the second condition, we can always deform $(\alpha_t)_{t\in[0,1]}$ with non-singular extremities to another such a path verifying $r_0 = 0 = r_{n+1}$: this is the main result of [18]. As most of the lemmas of [18] easily generalise to any critical index, we expect being able to deform any path $(\alpha_t, \xi_t)_{t\in[0,1]}$ with no singular extremities to another one verifying the hypothesis of theorem 3.1; even further, the non-trivial index rank should be shrinkable to two consecutive indexes $i, i+1$, as it was the case in the exact context.

References

[1] Abraham, R. and Marsden, J.E. – Foundations of mechanics, 2nd Ed., Revised, enlarged and reset, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, pp. xxii+m-xvi+806, 1978.

[2] Arnol′d, V. I. – Geometrical methods in the theory of ordinary differential equations, Grundlehren der Mathematischen Wissenschaften, vol. 250, 2nd Ed., Springer-Verlag, pp. xiv+351, 1988.

[3] Cerf, J. – La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie, Institut des Hautes Études Scientifiques. Publications Mathématiques, No. 39, pp. 5–173, 1970.

[4] Cerf, J. – Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = 0$), Lecture Notes in Mathematics, No. 53, Springer-Verlag, pp. xii+133, 1968.

[5] Farber, M. – Topology of closed one-forms, Mathematical Surveys and Monographs, vol. 108, American Mathematical Society, pp. xii+246, 2004.

[6] Hatcher, A. and Wagoner, J. – Pseudo-isotopies of compact manifolds. Part I, Astérisque, No. 6, Société Mathématique de France, Paris, pp. i+238, 1973.

[7] Hutchings, M. – Reidemeister torsion in generalized Morse theory, Forum Mathematicum, vol. 14, No. 2, pp. 209–244, 2002.

[8] Igusa, K. – What happens to Hatcher and Wagoner's formulas for $\pi_0 C(M)$ when the first Postnikov invariant of M is nontrivial?, Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), Lecture Notes in Math. vol. 1046, Springer-Verlag, pp. 104–172, 1984.

[9] Kinsey, L. C. – Pseudoisotopies and submersions of a compact manifold to the circle, Topology, vol. 26, No. 1, pp. 67–78, 1987.

[10] Latour, F. – Existence de 1-formes fermées non singulières dans une classe de cohomologie de de Rham, Institut des Hautes Études Scientifiques. Publications Mathématiques, No. 80, pp. 135–194 (1995), 1994.

[11] Laudenbach, F. and Blank, S. – Isotopie de formes fermées en dimension trois, Inventiones Mathematicae, vol. 54, No. 2, pp. 103–177, 1979.

[12] Laudenbach, F. – Submersions sur le cercle, Bulletin de la Société Mathématique de France, vol. 104, No. 4, pp. 417–431, 1976.

[13] Milnor, J. – Introduction to algebraic K-theory, Annals of Mathematics Studies, No. 72, Princeton University Press, pp. xiii+184, 1971.

[14] Milnor, J. – Lectures on the h-cobordism theorem, Notes by L. Siebenmann and J. Sondow, Princeton University Press, pp. v+116, 1965.

[15] Milnor, J. – Whitehead torsion, Bulletin of the American Mathematical Society, vol. 72, pp. 358–426, 1966.

[16] Moraga Ferrándiz, C. – An obstruction to isotopy of non-singular closed 1-forms, Work in progress.

[17] Moraga Ferrándiz, C. – Contribution à une théorie de Morse-Novikov à paramètre, PhD. Thesis, Université de Nantes, pp. xiv+123, 2012.
Available at http://hal.archives-ouvertes.fr/tel-00768575/

[18] Moraga Ferrándiz, C. – Elimination of extremal index zeroes from generic paths of closed 1-forms, Mathematische Zeitschrift, July 2014.

DOI: 10.1007/s00209-014-1332-4. Also available at arXiv: 1303.5918.

[19] Moraga Ferrándiz, C. – The s-cobordism theorem seen as a particular case of Latour's theorem, arXiv:1312.2564, 2013.

[20] Moser, J. – On the volume elements on a manifold, Transactions of the American Mathematical Society, vol. 120, pp. 286–294, 1965.

[21] Novikov, S.P. – The Hamiltonian formalism and a many-valued analogue of Morse theory, Russian Mathematical Surveys, vol. 37, No. 5, pp.1–56, 1982.

[22] Pajitnov, A.V. – Circle-valued Morse theory, de Gruyter Studies in Mathematics, vol. 32, Walter de Gruyter & Co., pp. x+454, 2006.

[23] Tischler, D. – On fibering certain foliated manifolds over S^1, Topology, Vol. 9, pp. 153–154, 1970.