Milnor invariants and the Homflypt Polynomial

(with J-B Meilhan, Univ. Grenoble I)

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Milnor Inv. (Milnor '54, '57)

L: n-comp. link $\subset S^3$

$G = \pi_1(S^3 \setminus L)$

G_k : k th lower central subgroup of G

$$\left(\text{i.e. } G_1 = G, \quad G_k = \underbrace{[G, G_{k-1}]}_{\substack{\text{generated by} \\ ghg^{-1}h^{-1},\ g \in G,\ h \in G_{k-1}}} \right)$$

G/G_k : generated by i th meridians α_i $(i = 1, \cdots, n)$

j th longitude l_j : written by $\alpha_1, \cdots, \alpha_n$

Magnus exp. of l_j :

$$l_j \xrightarrow[\alpha_i^{-1} = 1 - X_i + X_i^2 - \cdots \ \left(= \frac{1}{1+X_i}\right)]{\alpha_i = 1 + X_i} 1 + \sum \mu_L(i_1 \cdots i_s j) X_{i_1} \cdots X_{i_s}$$

(Here only $\mu_L(i_1 \cdots i_s j)$, $s < k$ is useful)

For a seq I of $\{1, \ldots, n\}$,

$$\Delta(I) = \mathrm{GCD}\{\mu_L(J) \mid J \xleftarrow{\hspace{2cm}} I\}$$

delete at least one term in I & permutate cyclicly

<u>Milnor Inv</u>

$\overline{\mu}_L(I)$: residue class of $\mu_L(I)$ mod $\Delta(I)$

<u>Remarks</u>

(1) $\overline{\mu}_L(I)$ is cobordism inv. for $\forall I$

(2) each number appear in I at most once

$\Rightarrow \overline{\mu}_L(I)$ is <u>link-homotopy</u> inv.

(We call this Milnor link-homotopy inv.)

(3) $i \neq j$, $\Delta(i, j) = 0$ &

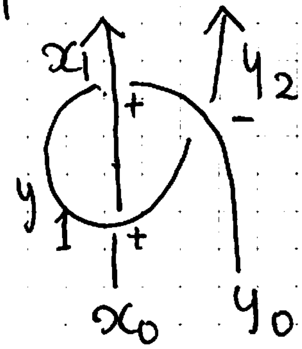
$\overline{\mu}_L(i, j)$: linking number of ith comp & jth comp.

((4) $\overline{\mu}_L(I)$ is <u>not</u> finite type inv. except for $I = ij$)

(5) Milnor inv shows that

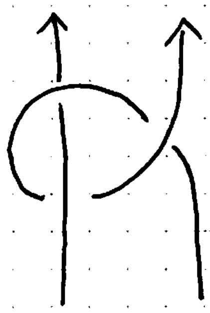
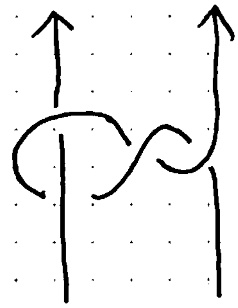
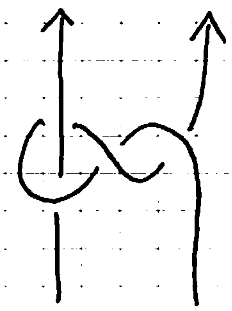
$\{2\text{-string link}\}/\overset{c}{\sim}$: non abelian
(cobordant)

Example:

$a =$ 

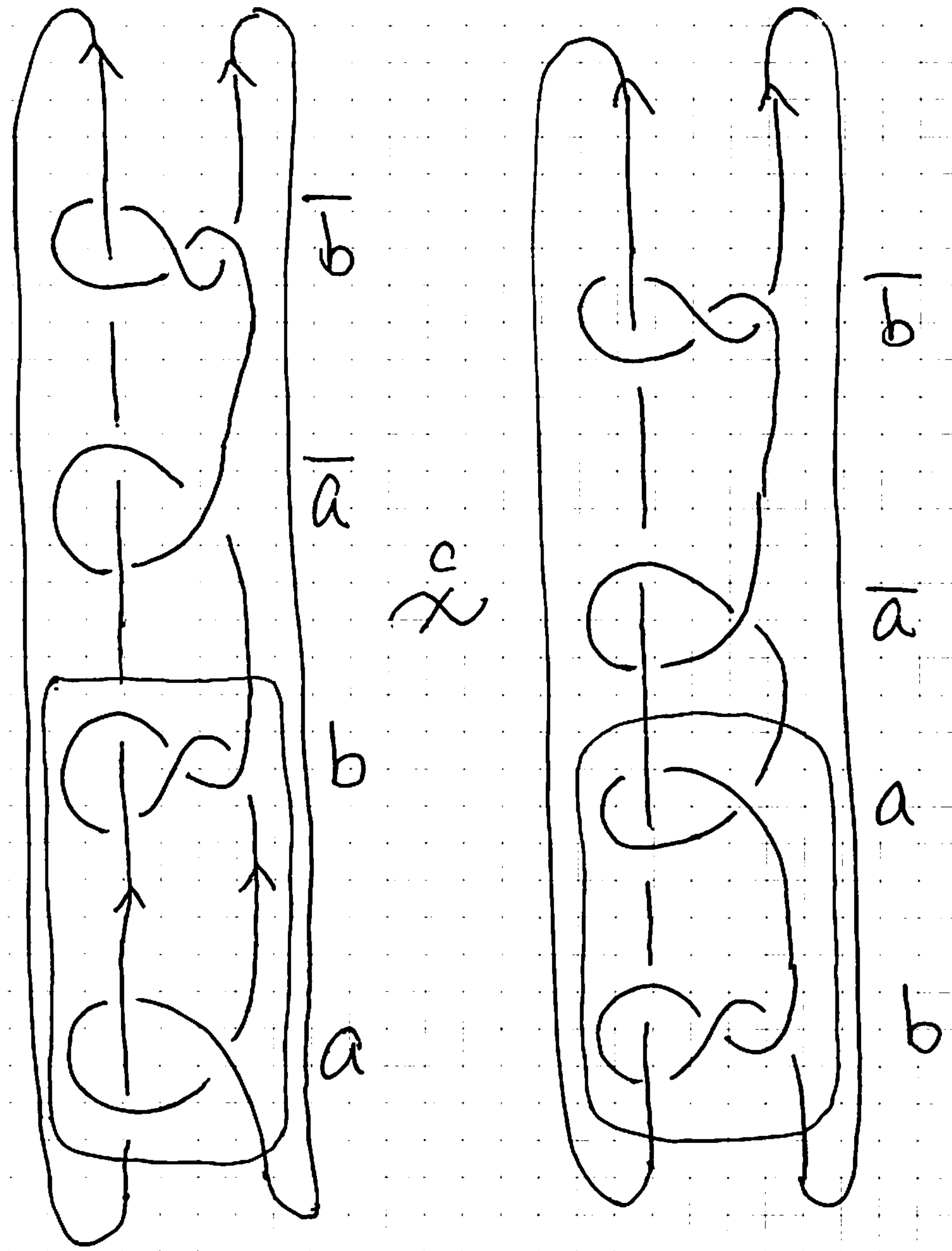
$l_x = y_1$

$l_y = x_1 y_2^{-1}$

$\bar{a} =$  $b =$  $\bar{b} =$ 

- $ab\bar{a}\bar{b}$ has nontrivial Milnor inv. of length 9 (length ≤ 8 vanish)
- $ba\bar{a}\bar{b} \overset{c}{\sim} \uparrow\uparrow \Rightarrow$ All Milnor inv $= 0$

$\therefore \quad ab\bar{a}\bar{b} \overset{c}{\not\sim} ba\bar{a}\bar{b}$


$\bar{b}$
$\bar{a}$
b
a
$\overset{c}{\sim}$
$\bar{b}$
$\bar{a}$
a
b

Relations between Alexander-Conway Poly & Milnor Inv.

L : n-comp. link with $\bar{\mu}_L(I)=0$
for any seq. I of length $\leq k$

$\Rightarrow$

Conway Poly $\nabla_L(z)$ is divisible by $z^{k(n-1)}$
&
the coefficient of $z^{k(n-1)}$ = $\det(a_{ij})$,

$$a_{ij} = \begin{cases} \displaystyle\sum_{i_1,\dots,i_{k-1}} \bar{\mu}_L(i\, i_1 \cdots i_{k-1}\, j) & (k \geq 2) \\[2ex] = \begin{cases} \bar{\mu}_L(ij) & i \neq j \\ -\displaystyle\sum_{r \neq i} \bar{\mu}_L(ir) & i = j \end{cases} & (k=1) \end{cases}$$

$(1 \leq i, j \leq n-1)$

(Hoste '85 for $k=1$,
Cochran '85 for $k=2$, $n=3$
Levine '99 for $k \geq 2$)

Closure knot for L

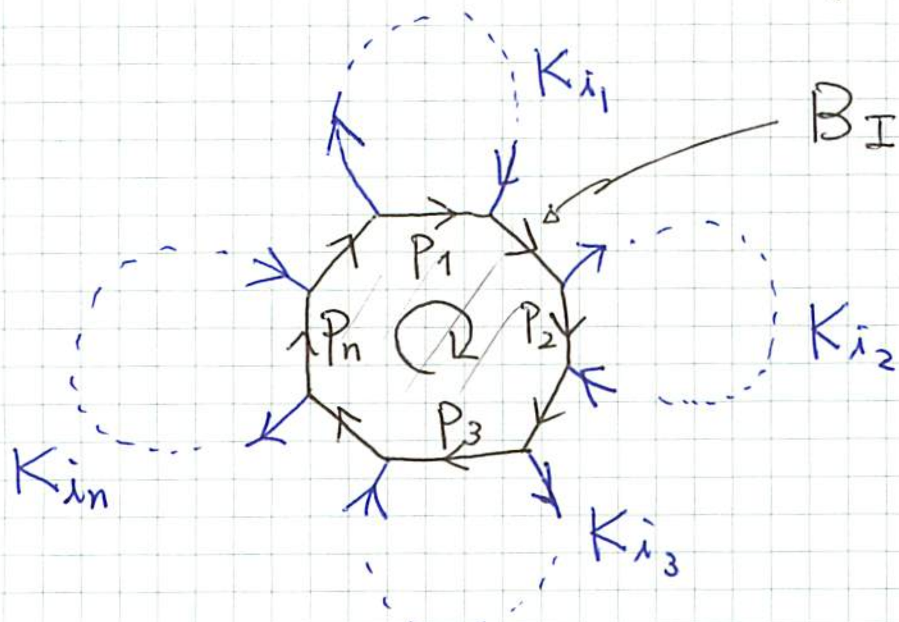
(We will mention for only Milnor link-homotopy inv. case)

$L = K_1 \cup \cdots \cup K_n$: n-comp. link

$I = i_1 \cdots i_n$: seq. s.t. $\{i_1 \cdots i_n\} = \{1, \dots, n\}$

I-fusion disk B_I : $2n$-gon with nonadjacent edges $P_1, \dots, P_n$

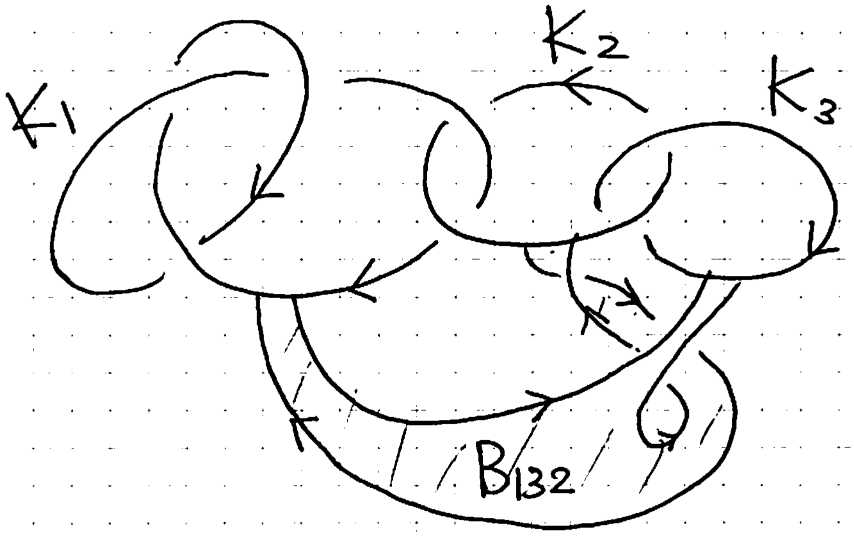
s.t.
$$\begin{cases} B_I \cap L = P_1 \cup \cdots \cup P_n, \\ B_I \cap K_{i_j} = P_j, \\ \text{ori } \partial B_I \text{ \& ori } K_{i_j} \text{ opposite} \end{cases}$$

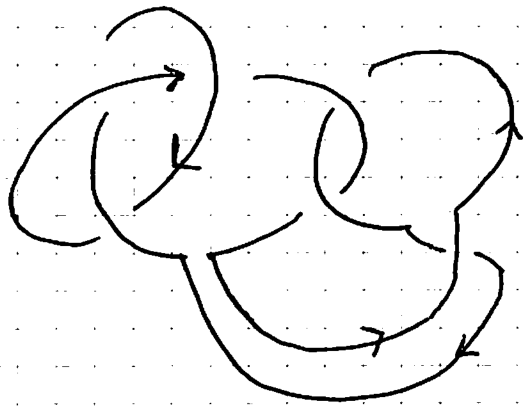


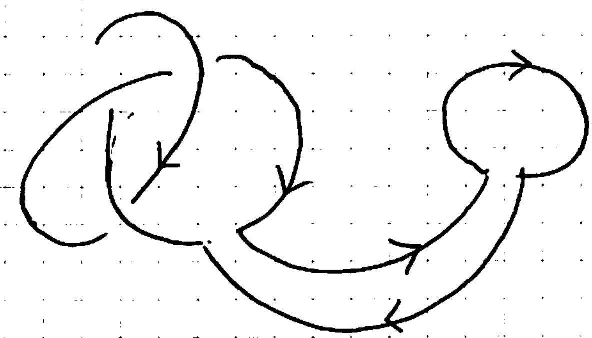
J : subseq. of I,

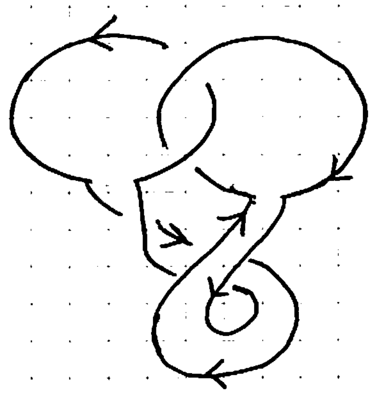
$$L_J = \overline{\left(\left(\bigcup_{i \in J} K_i\right) \cup \partial B_I\right) \setminus \left(\left(\bigcup_{i \in J} K_i\right) \cap \partial B_I\right)}$$

Example $I = 132$



$L_{12} =$ 

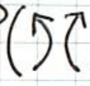
$L_{13} =$ 

$L_{23} =$ 

Homflypt poly

$P(L;t,z) \in \mathbb{Z}[t^{\pm}, z^{\pm}]$: two variable poly defined by

(1) $P(\text{trivial knot}; t, z) = 1$

(2) $t^{-1} P($$; t, z) - t P($$; t, z) = z P($$; t, z)$

For n-comp. link L,

$$P(L;t,z) = \sum_{k=1}^{N} \underbrace{P_{2k-1-n}(L;t)}_{\text{poly of } t} z^{2k-1-n}$$

In particular, for $n=1$,

$$P(L;t,z) = \sum_{k=1}^{N} P_{2k-2}(L;t) z^{2k-2}$$

We use $P_0(L;t)$ (remark : $P_0(L;1) = 1$)

Let $P_0^{(m)}(L) = (P_0(L;t))^{(m)}_{|t=1}$

$$(\log P_0(L))^{(m)} = (\log P_0(L;t))^{(m)}_{|t=1}$$

Theorem

L : n-comp. link ($n \geq 3$) with length $\leq \underline{\underline{k}}$
Milnor link-homotopy inv. vanishing

$\Rightarrow$ $\forall I$: seq. of ($\underline{\underline{m}}+1$) distinct numbers
($3 \leq m+1 \leq 2\underline{\underline{k}}+1$)

$\forall I$-fusion disk,

$$\bar{\mu}_L(I) \equiv \frac{-1}{\underline{\underline{m}}!\, 2^{\underline{\underline{m}}}} \sum_{\substack{J:\text{subseq.}\\ \text{of } I}} (-1)^{|J|} \left(\log P_0(L_J)\right)^{(\underline{\underline{m}})}$$

$$(\bmod\ \Delta(I))$$

(where
$|J|$: length of J)

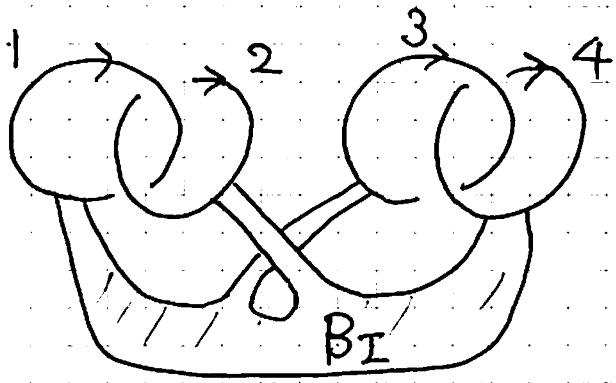
In particular, for $m+1 = k+1$ (≥ 3)

$$\bar{\mu}_L(I) = \frac{-1}{k!\, 2^k} \sum_{\substack{J:\text{subseq.}\\ \text{of } I}} (-1)^{|J|} P_0^{(k)}(L_J)$$

$$(\in \mathbb{Z})$$

Remarks

(1) assumption of vanishing Milnor inv of length $\leqslant k$ is essential



$I = 1324$

$\bar{\mu}(12) = \bar{\mu}(34) = 1$

$\therefore \Delta(I) = 1$ &

$\bar{\mu}(I) \equiv 0$

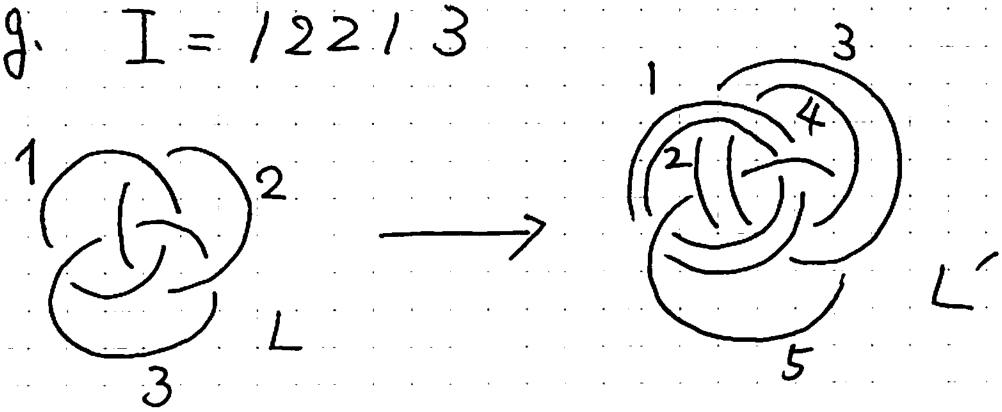
$$\sum_{\substack{J:\text{subseq}\\ \text{of } 1324}} (-1)^{|J|} (\log P_0(L_J))^{(3)} = 24$$

$$3!\,2^3 \neq 24$$

(2) I contains index which appears at least twice

$\Rightarrow \bar{\mu}_L(I)$ is given by Milnor link homotopy inv. of a "parallel" of L

e.g. $I = 12213$



$$\bar{\mu}_L(12213) \equiv \bar{\mu}_{L'}(13425)$$

Direction of Proof

$I = 12 \cdots n$

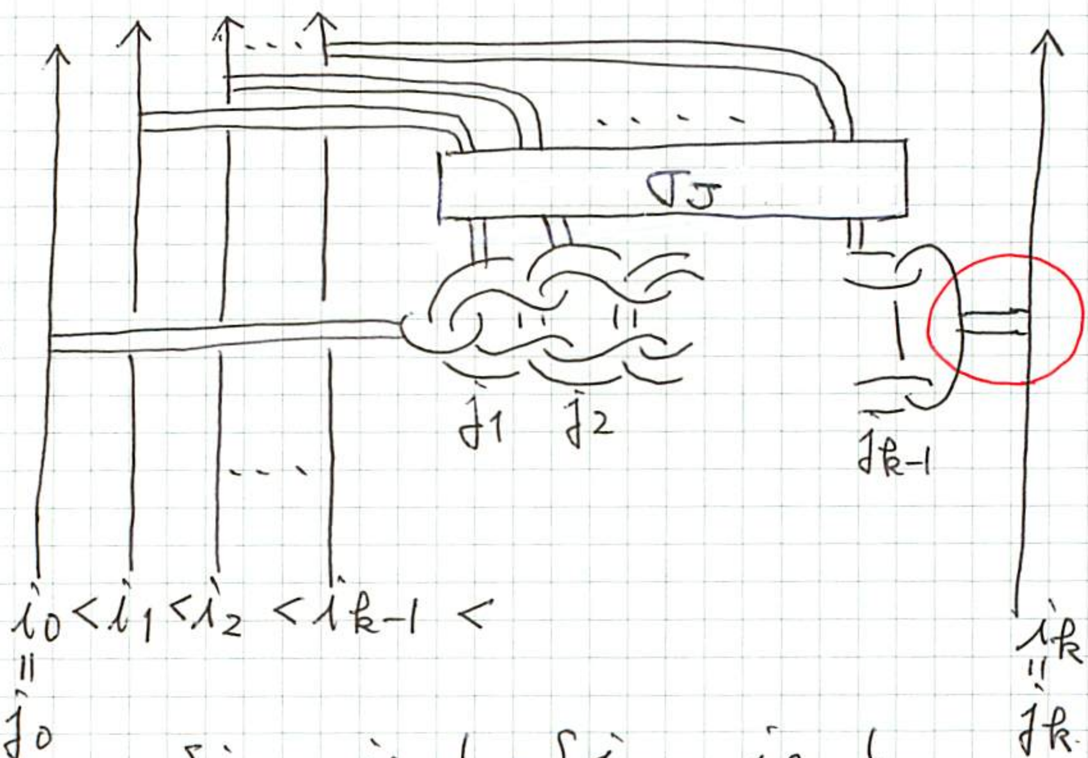
L : n-comp. link ,

$L \cup B_I \rightsquigarrow$ ℓ : string link $\subset D^2 \times I$

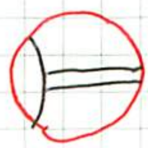
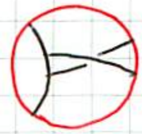
consider the complement of reg. nbd of B_I

$$\ell \overset{\ell.h}{\sim} \ell_1 \times \cdots \times \ell_{n-1}, \quad \ell_k = \prod_{J \in \mathcal{J}_k} V_J^{x_J}$$

$\mathcal{J}_k$: set of all seq. $j_0 < j_1, \cdots, j_{k-1} < j_k$ of $(k+1)$ distinct numbers

$V_J =$ 

$\{i_1, \cdots, i_{k-1}\} = \{j_1 \cdots j_{k-1}\}$

$V_J^{-1} = V_J$ with replaced  with 

(Note: J : subseq of $I = 12 \cdots n \Rightarrow \boxed{\sigma_J} = \boxed{|||\cdots|||}$)

Remarks

(1) $\bar{\mu}_L(I) \equiv \mu_\ell(I) \mod \Delta_L(I) = \Delta_\ell(I)$

(2) By the assumption of vanishing Milnor link-homotopy inv.

$\forall J$ with length $\leq \frac{n}{2}$, $\quad x_J = 0$

We show the following:

(i) $\mu_\ell(I) \equiv x_I \mod \mathrm{GCD}\left\{ x_J \,\middle|\, \begin{array}{l} J: \text{subseq} \\ \text{of } I, \\ J \neq I \end{array} \right\}$

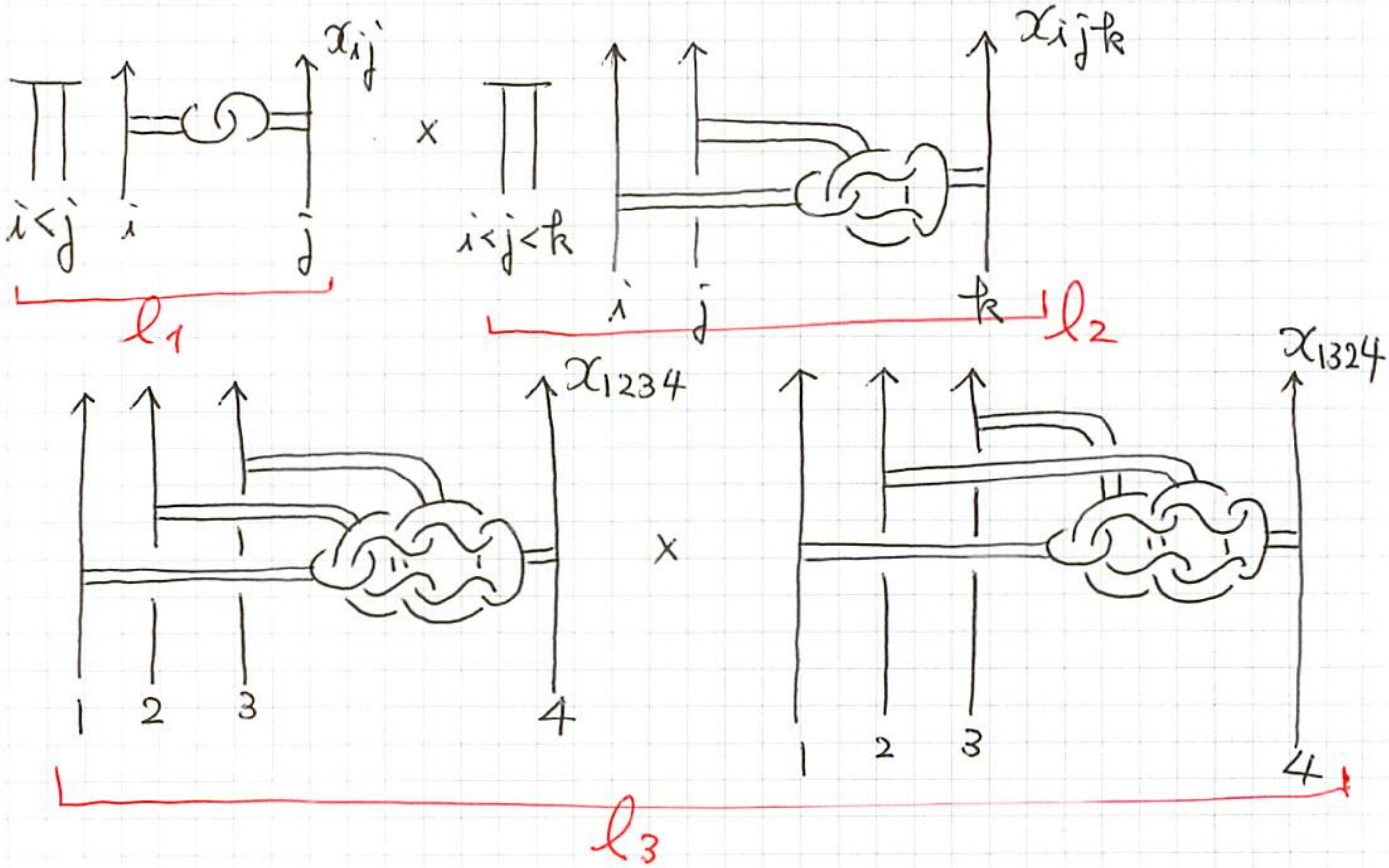
&

$$\Delta_\ell(I) = \mathrm{GCD}\left\{ x_J \,\middle|\, \begin{array}{l} J: \text{subseq of } I \\ J \neq I \end{array} \right\}$$

(ii) $\dfrac{-1}{m!\,2^m} \displaystyle\sum_J (-1)^{|J|} \left(\log P_0(L_J) \right)^{(m)} \equiv x_I$

$\mod \mathrm{GCD}\left\{ x_J \,\middle|\, \begin{array}{l} J: \text{subseq of } I \\ J \neq I \end{array} \right\}$

Example 4-string link $\ell \overset{l.h}{\sim} \ell_1 \times \ell_2 \times \ell_3$



$$\mu_\ell(1234) \equiv x_{1234} \quad \mathrm{GCD}\left\{ \begin{array}{l} x_{12},\ x_{13},\ x_{14},\ x_{23},\ x_{24} \\ x_{34},\ x_{123},\ x_{124},\ x_{134}, \\ x_{234} \end{array} \right\}$$