Total Curvature of Graphs after Milnor and Euler

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Definitions of Total Curvature

Γ: piecewise-smooth embedded spatial finite graph in $\mathbf{R}^3$.

Total Curvature

$$\mathcal{C}(\Gamma) = \sum_{i=1}^{N} c(q_i) + \int_{\Gamma_{\mathrm{reg}}} |k| ds$$

Γ_{reg}: the smooth part, k: geodesic curvature vector
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Theorem (Fenchel 1929, Milnor Fary 1950)

Let $c(q_i)$ be the external angle at q_i.
Then $\mathcal{C}(\Gamma) \geq 2\pi$ and "$=$" iff Γ is planar, convex.
$\mathcal{C}(\Gamma) \leq 4\pi$ implies Γ is unknotted.

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Cone Total Curvature (Gulliver-Y.)

$$c(q) = \mathrm{ctc}(q) := \sup_{e \in S^2} \sum_{i=1}^{d} \left(\frac{\pi}{2} - \arccos\langle T_i, e\rangle\right)$$

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Definition

Define net total curvature of Γ by

$$\mathrm{NTC}(\Gamma) = \sum_{i=1}^{N} \mathrm{ntc}(q_i) + \int_{\Gamma_{\mathrm{reg}}} |k| ds$$

Crofton-like formula for NTC

Theorem 1 (Gulliver-Y.)

$$\mathrm{NTC}(\Gamma) = \frac{1}{2}\int_{S^2} \mu(e)\, dA_{S^2}(e)$$

where

$$\mu(e) = \sum_q \{\mathrm{nlm}^+(e, q) : \ q \text{ a vertex or a critical point of } \langle e, \cdot \rangle\}$$

and where

$$\mathrm{nlm}(e, q) = \frac{1}{2}[\mathrm{lmax}(\Gamma')(e, q) - \mathrm{lmin}(\Gamma')(e, q)].$$

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- $\sum_{i=1}^{d} \chi_i(e) = d^-(e, q) - d^+(e, q)$
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  which implies
  $\mathrm{ntc}(q) = \frac{1}{2} \int_{S^2} [\mathrm{nlm}(e, q)]^+ dA_{S^2}$.
- $\#\{p \in \Gamma : \langle e, p\rangle = s_0\} = 2 \sum_q \{\mathrm{nlm}(e, q) : \langle e, q\rangle > s_0\}$
  (cf. knot (Milnor))
- When $f : \Gamma \to \mathbf{R}^3$ not necessarily imbedding, NTC is defined using RHS of the formula. (cf. crossing can be dealt with.)

Crofton-like formula for Continuous Graphs

Theorem 2 (G.-Y.)

Γ: continuous graph, embedded in $\mathbf{R}^3$. Then

$$\mathrm{NTC}(\Gamma) = \frac{1}{2}\int_{S^2} \mu(e)\, dA_{S^2}(e)$$

where

$$\mathrm{NTC}(\Gamma) = \sup_P \mathrm{NTC}(P)$$

P : Γ-approximating polygonal graph and where

$$\mu(e) := \lim_{k\to\infty} \mu_{P_k}(e)$$

where P_k is a sequences of Γ-approximating graphs suitably refined for e.

Remark P' refines $P \Rightarrow \mathrm{NTC}(P) \leq \mathrm{NTC}(P')$.

Corollary (G.-Y.)

Γ: continuous graph, with $\mathrm{NTC}(\Gamma) < \infty$. Then Γ is tame, i.e. isotopic to a polyhedral graph.

Remark $\exists$ tame graphs with $\mathrm{NTC}(\Gamma) = \infty$.

infimum realizing embeddings/immersions

Define
(1)$\mathrm{NTC}(\{\Gamma\}) = \inf_{f:\Gamma\to\mathbf{R}^3} \mathrm{NTC}(f)$ and
(2)$\mathrm{NTC}([\Gamma]) = \inf_{f:\Gamma\to\mathbf{R}^3} \mathrm{NTC}(f)$ for an isotopy class $[\Gamma]$ of embeddings into $\mathbf{R}^3$.

Theorem 3 (G.-Y.)

(1) $\mathrm{NTC}(\{\Gamma\})$ is assumed by a mapping $f_0 : \Gamma \to \mathbf{R}$.
(2) $\mathrm{NTC}([\Gamma])$ is assumed by a mapping $f_1 : \Gamma \to \mathbf{R} \subset \mathbf{R}^3$ in the closure of the given isotopy class.
(3) If $f_1 : \Gamma \to \mathbf{R} \subset \mathbf{R}^3$ is in the closure of the given isotopy class with $\mathrm{NTC}(f_1) = \mathrm{NTC}([\Gamma])$, then for any $\delta > 0$ there is an embedding $f : \Gamma \to \mathbf{R}^3$ with $\mathrm{NTC}(f) \leq \mathrm{NTC}(f_1) + \delta$.

Trivalent Graphs

Define $B(f) := \frac{1}{2}\,\#\{\text{local extrema}\}$ to be extended bridge number of $f : \Gamma \to \mathbf{R}^3$.

Theorem 4 (G.-Y.)

A trivalent Γ has $\mathrm{NTC}(\Gamma) = \frac{1}{2}C(\Gamma')$ where Γ' is an Euler circuit of the double of Γ. Also we have $\mathrm{NTC}(\{\Gamma\}) = \pi(2B(\{\Gamma\}) + \frac{k}{2})$ and $\mathrm{NTC}([\Gamma]) = \pi(2B([\Gamma]) + \frac{k}{2})$.

Note a trivalent graph Γ with k vertices has $\chi(\Gamma) = -k/2$.
Example: Γ^* the dual graph of one skelton of a triangulation of S^2. Koebe-Andreev-Thurston says $\exists$ a circle packing with the centers $= V(\Gamma^*)$. Then the circle-packing induced Γ^* has $B(\Gamma^*) = 1$ and $\mathrm{NTC}(\Gamma^*) = \pi(2 - \chi(\Gamma^*))$.

Lowerbounds

List of Lower Bounds

$\mathrm{NTC}(\{W_m\}) = \pi(2 + \lceil \frac{m}{2} \rceil)$
$\mathrm{NTC}(\{K_{2\ell}\}) = \pi\ell^2$
$\mathrm{NTC}(\{K_{2\ell+1}\}) = \pi\ell(\ell + 1)$
$\mathrm{NTC}(\{K_{m,n}\}) = \pi\lceil \frac{mn}{2} \rceil$
$\mathrm{NTC}(\{\theta_m\}) = m\pi$. $(\theta_m \simeq K_{m,2})$

Theorem 5 (G.-Y.)

$f : \theta \to \mathbf{R}^3$ continuous embedding. Let $\Gamma = f(\theta)$
Then $\mathrm{NTC}(\Gamma) \geq 3\pi$ with "$=$" iff Γ is planar, convex curve plus a straight chord.
And if $\mathrm{NTC}(\Gamma) < 4\pi$, then Γ is standard.

Gulliver-Yamada arxiv:1101.2305.
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