Every link has a (3,4)-diagram

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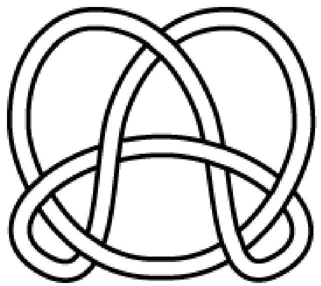
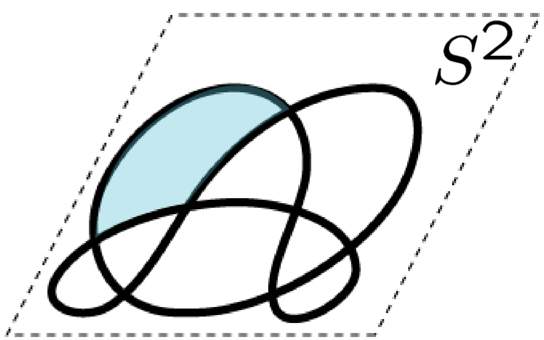
Complementary regions of a diagram



$\subset S^3$

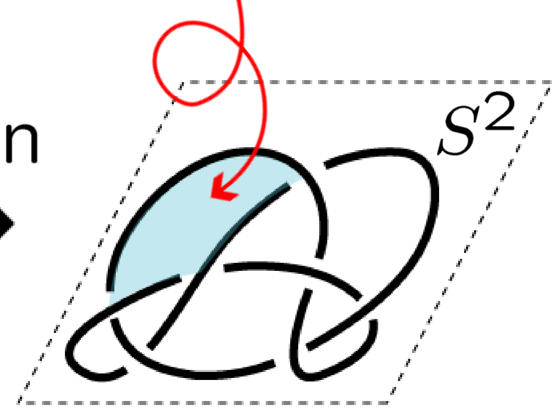
K:knot(1-comp.link)

projection



crossing infomation

a face of D

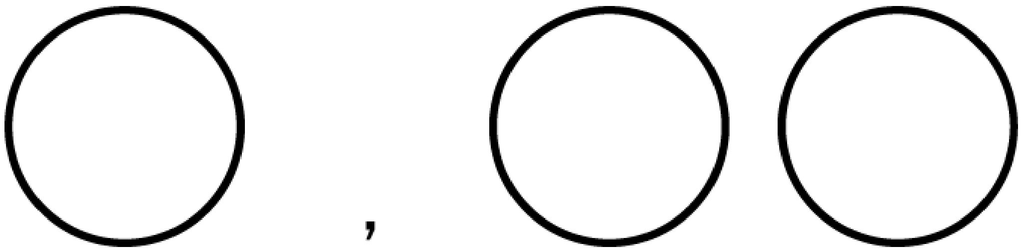


P:projection of K

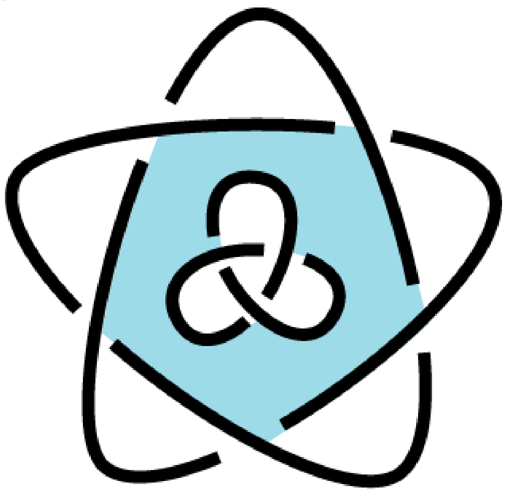
D:diagram of K

Assumptions

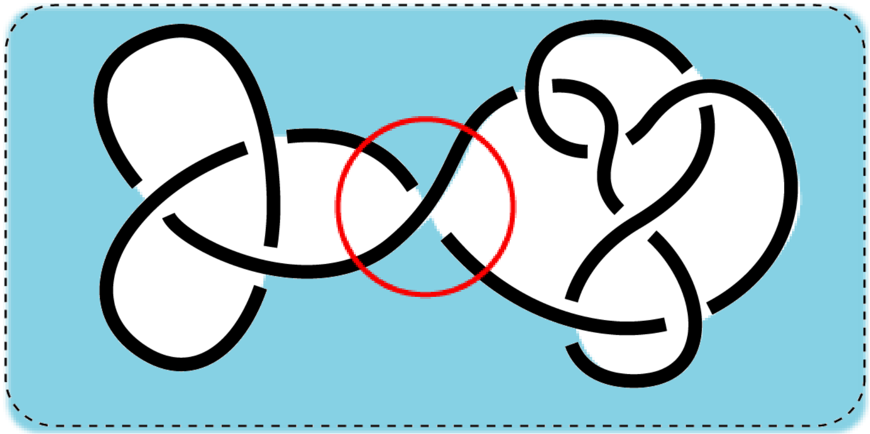
1) We do not consider diagrams without a crosssing.



2) Diagrams are connected and reduced.



not connected



not reduced

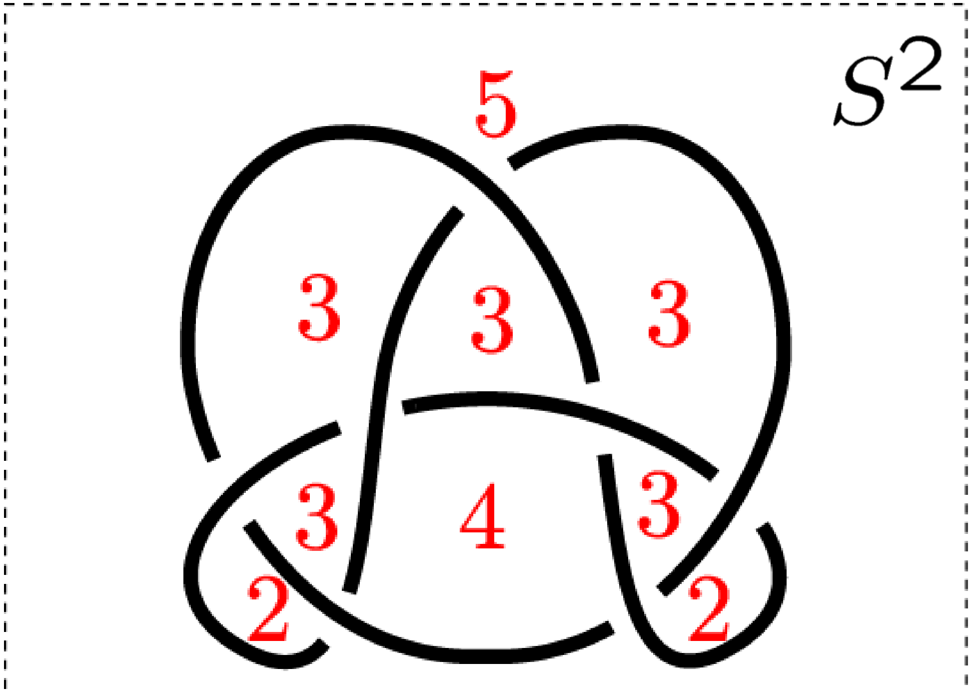
Notations

D: a diagram of a link

$$p_n(D) := \#\text{ of } n\text{-sided regions of } D$$

$$p_{\text{odd}}(D) := \sum_{n\,:\,\text{odd}} p_n(D)$$

Example



D : a diagram of a link

$$p_n(D) = \begin{cases} 2 & (n = 2) \\ 5 & (n = 3) \\ 1 & (n = 4) \\ 1 & (n = 5) \\ 0 & (\text{otherwise}) \end{cases}$$

$$p_{\mathrm{odd}}(D) = 6$$

Implication from Euler's formula

D : a link diagram (4-regular graph of S^2)

By Euler's formula $v - e + f = 2$, we have

$$2p_2(D) + p_3(D) = 8 + p_5(D) + 2p_6(D) + 3p_7(D) + \cdots.$$

Easy consequence

1) $p_2(D) \neq 0$ or $p_3(D) \neq 0$
2) $p_{\mathrm{odd}}(D)$ is even.

Known results about 4-valent graphs in S^2

D : 4-regular simple graph on S^2

(link diagram s.t. $p_2(D) = 0$)

$$p_3 = 8 + p_5 + 2p_6 + 3p_7 + \cdots \quad (*)$$

Theorem (Grunbaum 1969)

$\forall p_n \in N (n \neq 4)$ satisfying $(*)$,

$\exists p_4 \in N$,

$\exists$ a knot diagram D

s.t. $p_n(D) = p_n \ (n \geq 3)$

Theorem (Jeöng 1995)

$\forall p_n \in N (n \neq 4)$ satisfying $(*)$,

$\exists p_4 \in N$,

$\exists$ a knot diagram D

s.t. $p_n(D) = p_n \ (n \geq 3)$

Theorem (Enns 1982)

$p_3 = 8$, $p_n = 0 \ (n \geq 5)$, $\forall p_4 \in N$,

$\exists$ a link diagram D

s.t. $p_n(D) = p_n \ (n \geq 3)$

$(a_1, a_2, a_3, \ldots)$-diagram

$(a_1, a_2, a_3, \ldots)$: a strictly increasing sequence of integers

(finite/infinite, $a_1 \geq 2$)

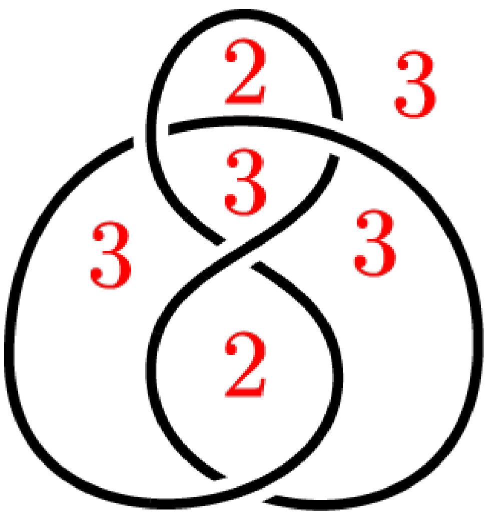
Definition 1

D : $(a_1, a_2, a_3, \ldots)$-diagram

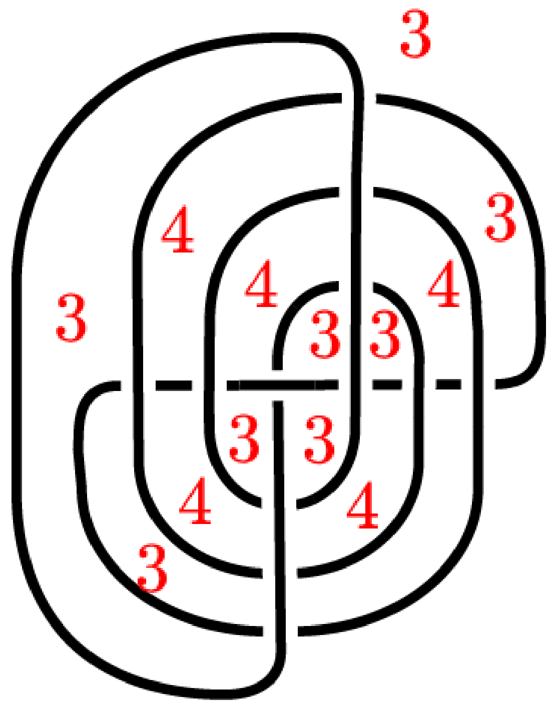
$\Longleftrightarrow$ Each face of D is an a_n-gon for some n.

Example (diagrams of a figure eight knot)

(2, 3, 4)-diagram, (2, 3, 4, 5)-diagram, . . .



(2, 3)-diagram



(3, 4)-diagram

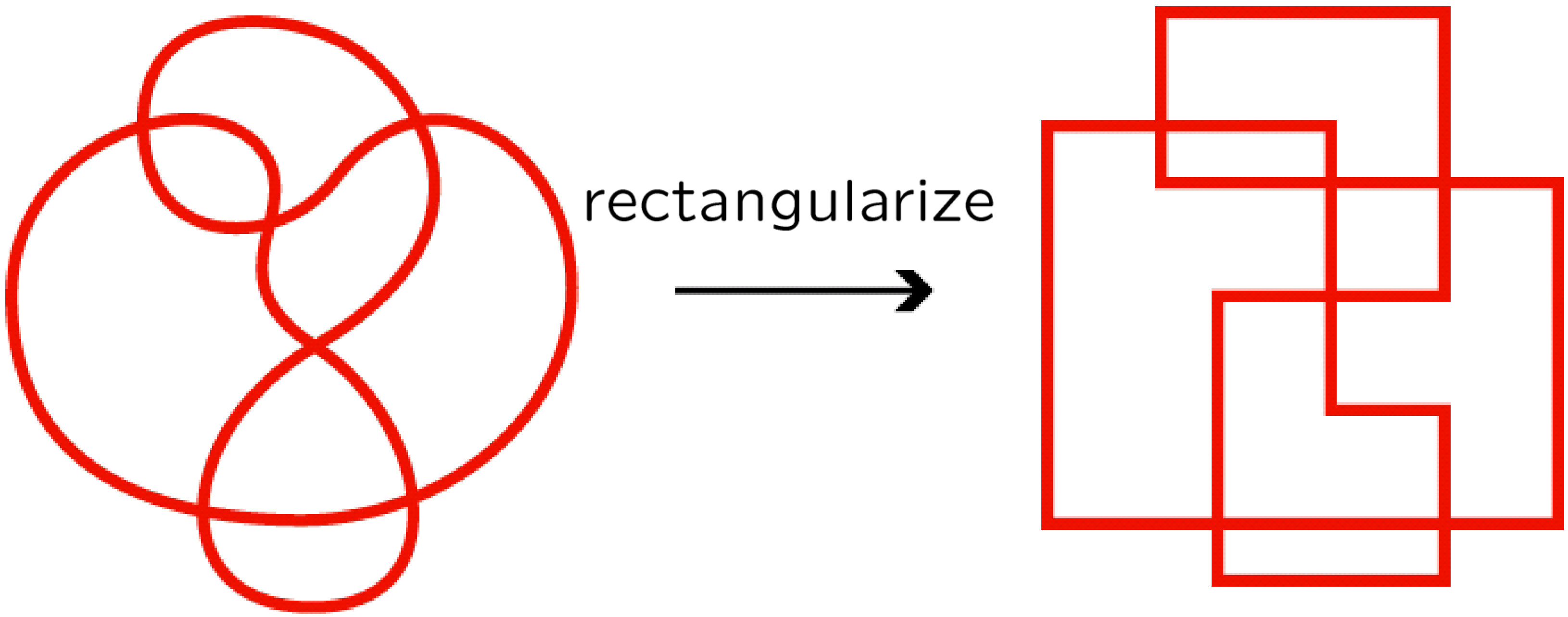
Theorem [A-S-T]

1) Every link has a $(2, 4, 5)$-diagram.

2) Every link has a $(3, 4, n)$-diagram $(n \geq 5)$.

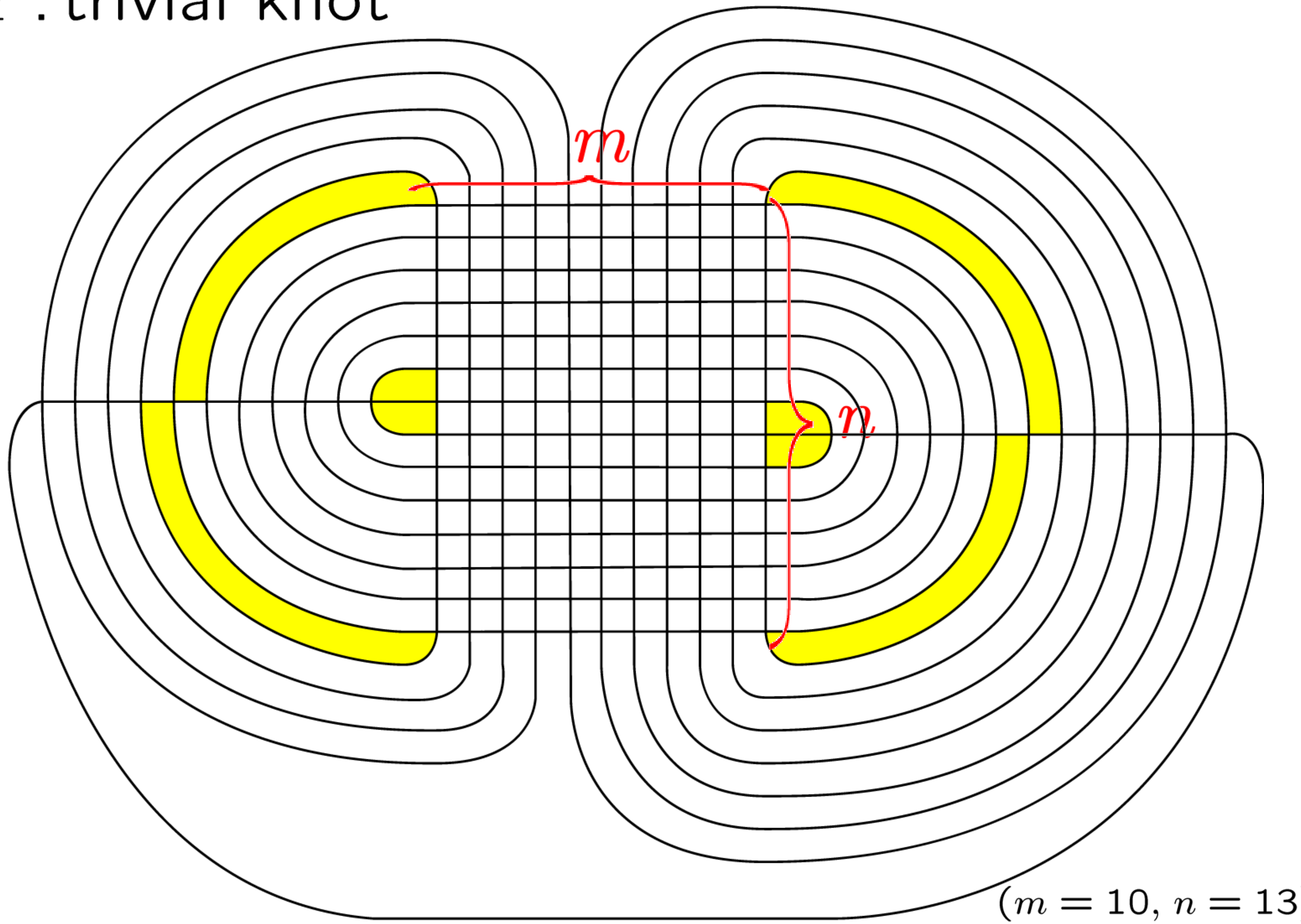
Proof of (1) (n=5)

L : link

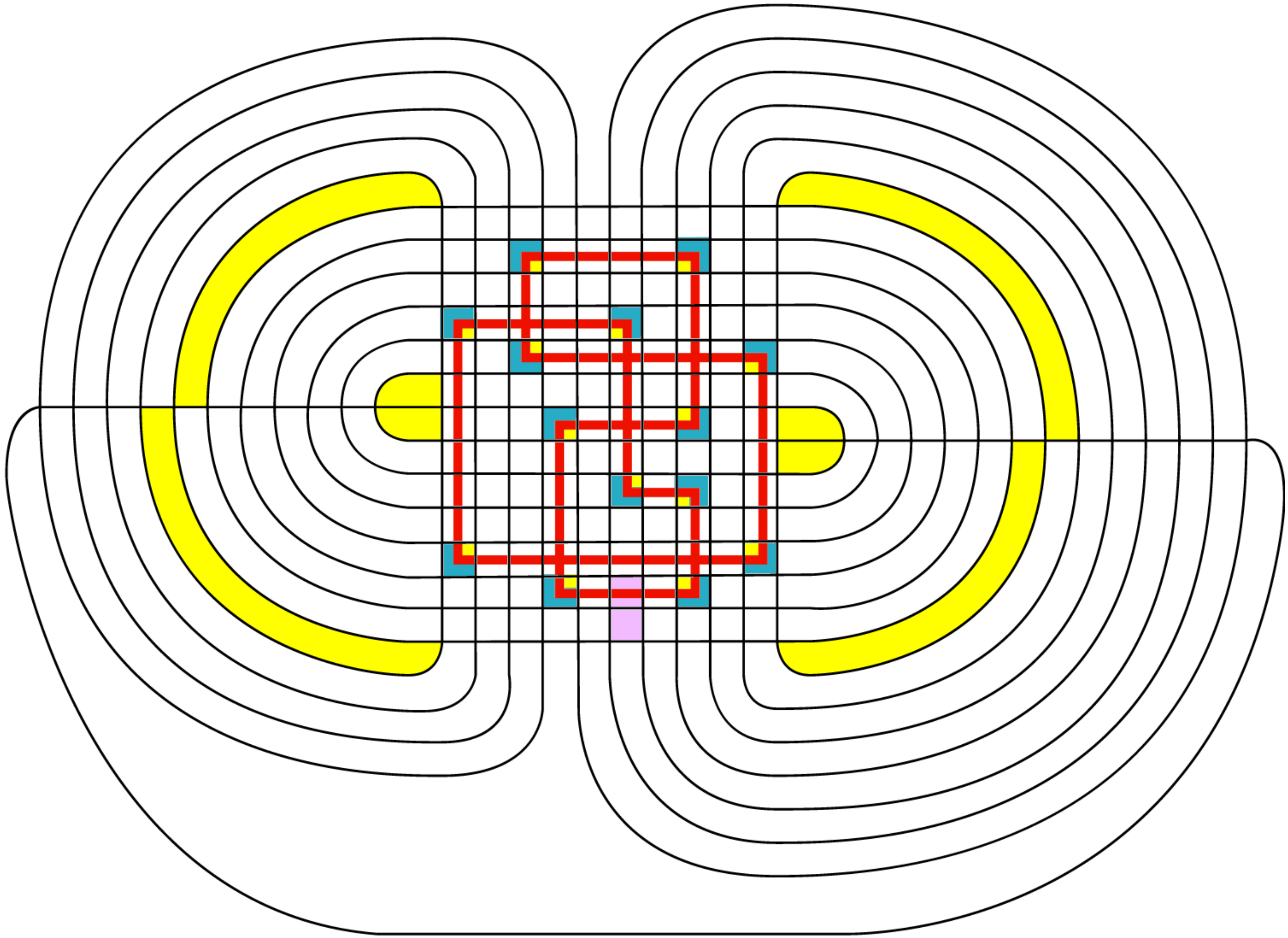


D : diagram of L

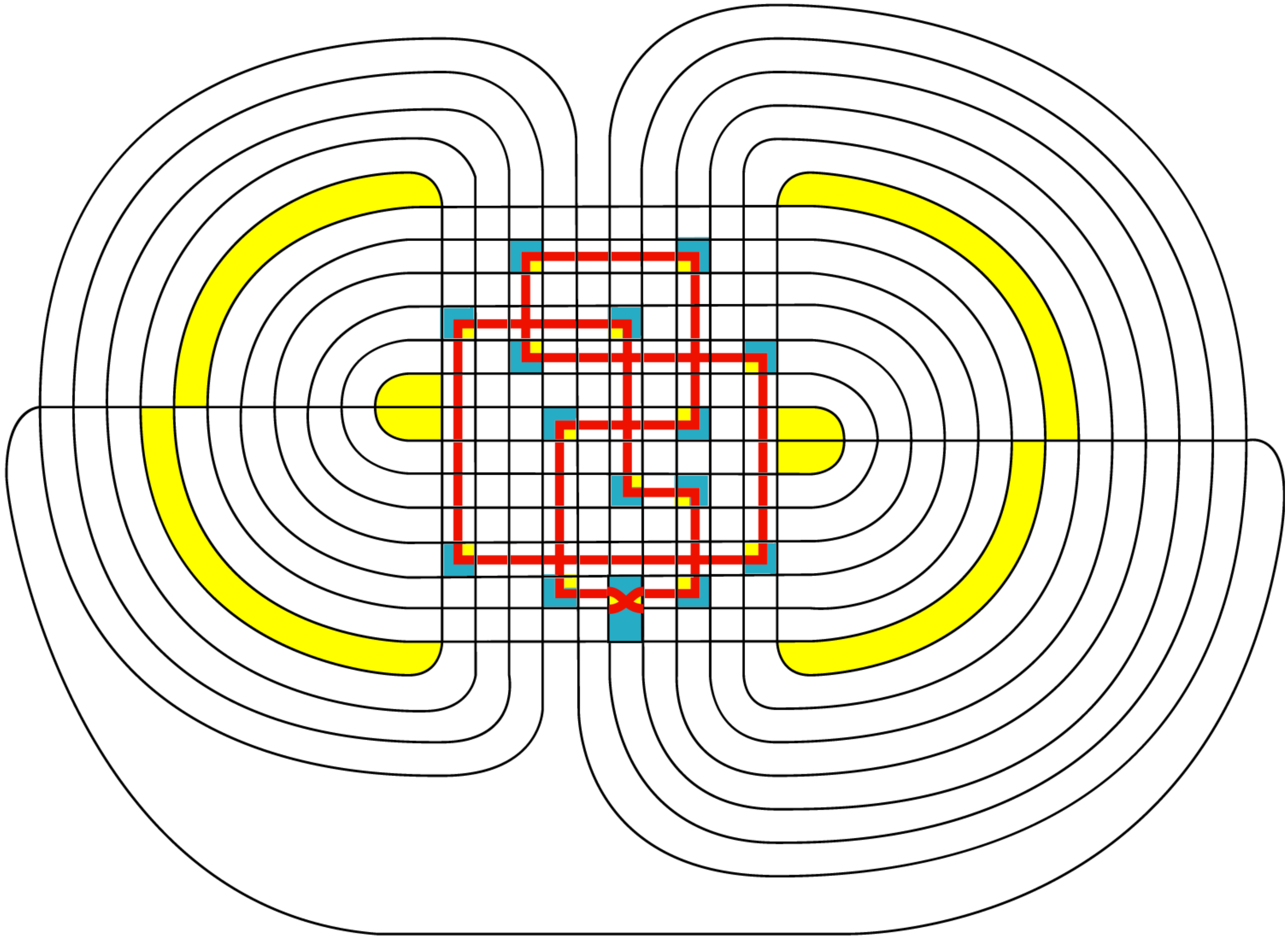
T : trivial knot



$Q(m, n)$: (3,4)-diagram of T



(3,4,5)-diagram of $L \bigsqcup T$

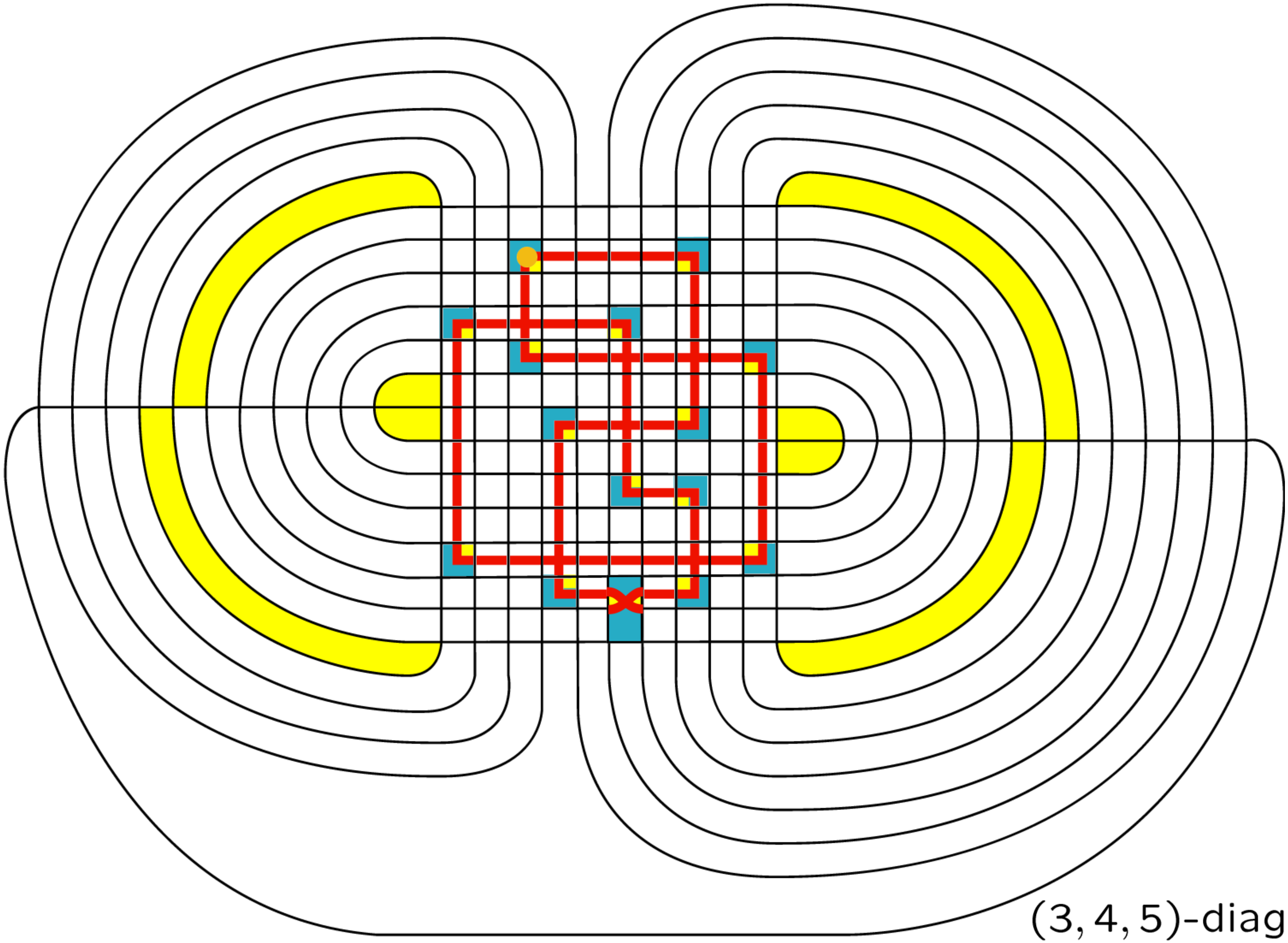


(3,4,5)-diagram of $L \sharp T$

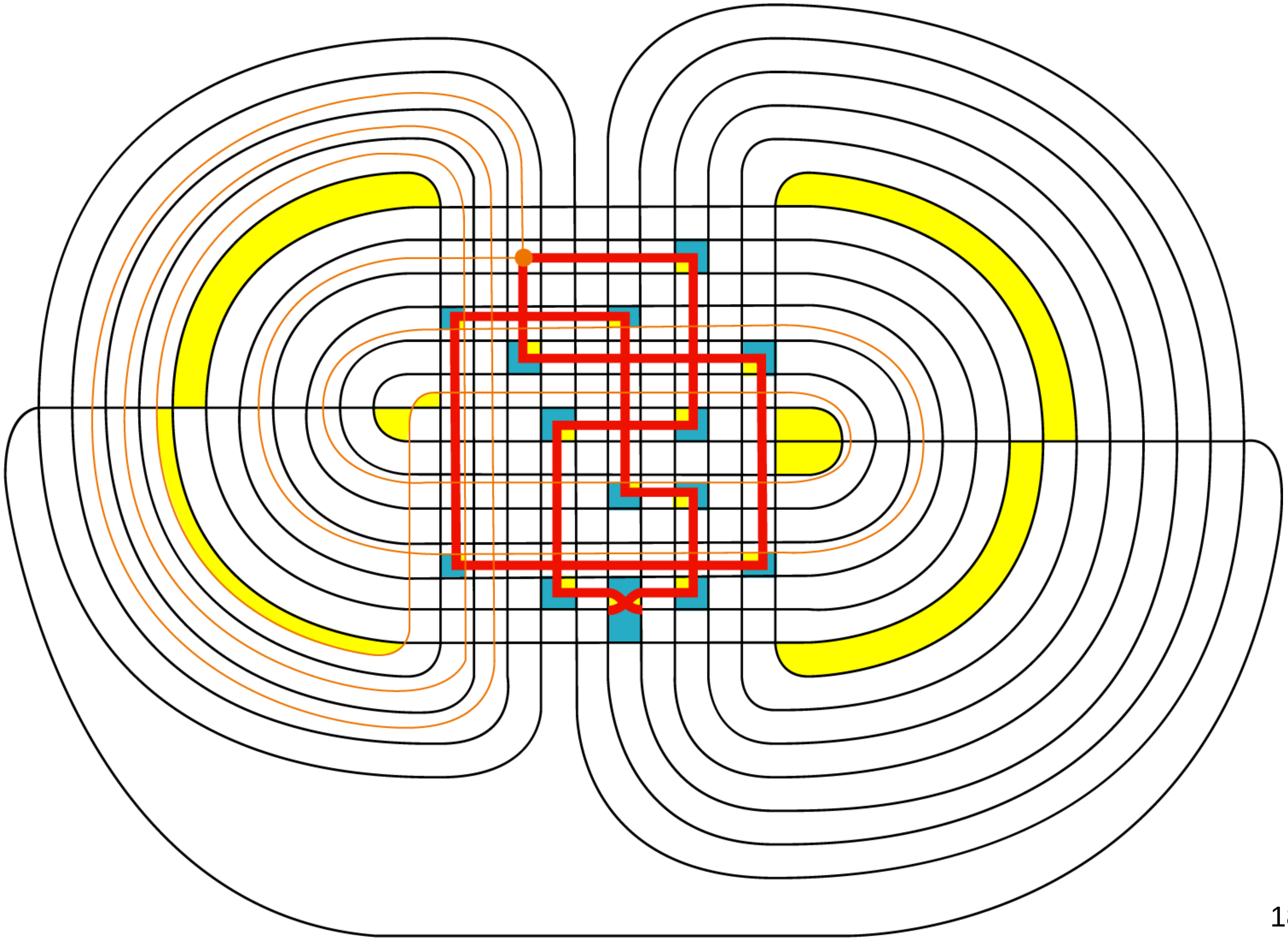
Main Theorem

Every link has a $(3,4)$-diagram.

Idea of the proof

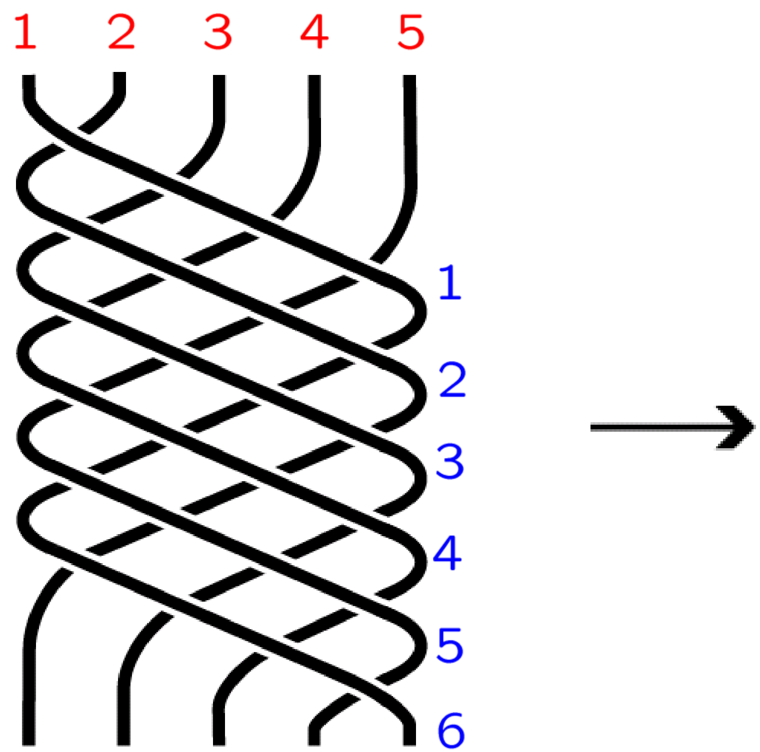


(3, 4, 5)-diagram

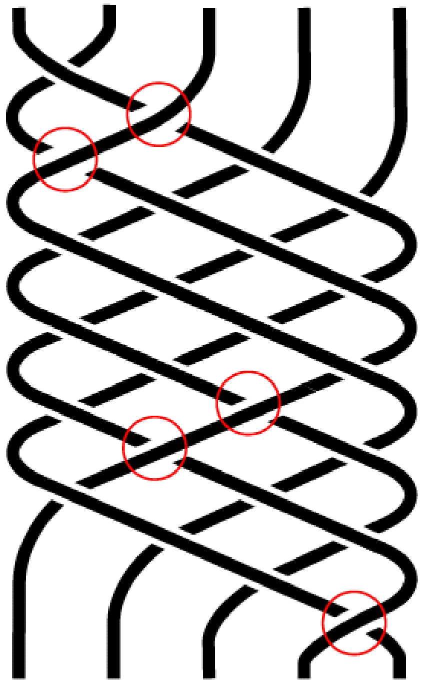


Sketch proof

$(\sigma_1, \sigma_2, \ldots, \sigma_{p-1})^q$: toric braid of type (p, q)



toric braid of type(5,6)

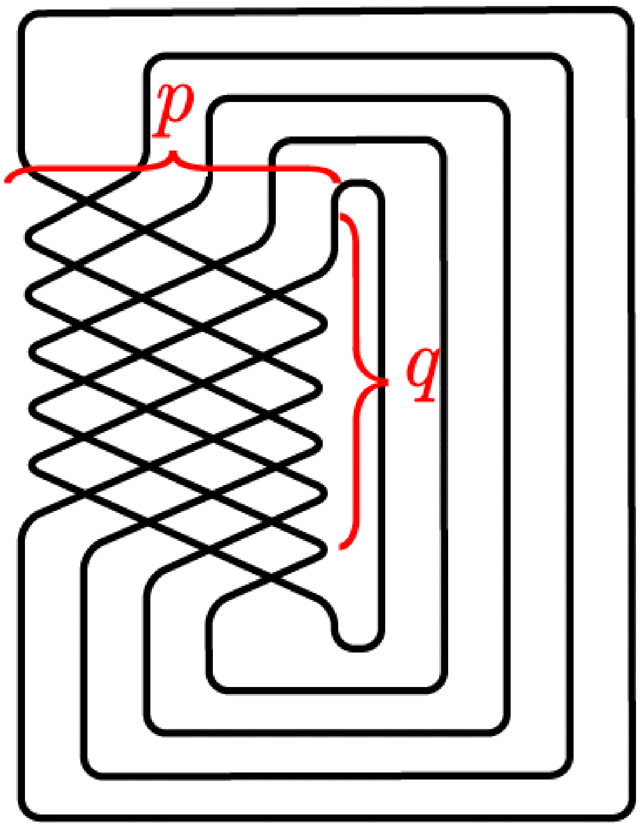


quasitoric braid of type (5,6)

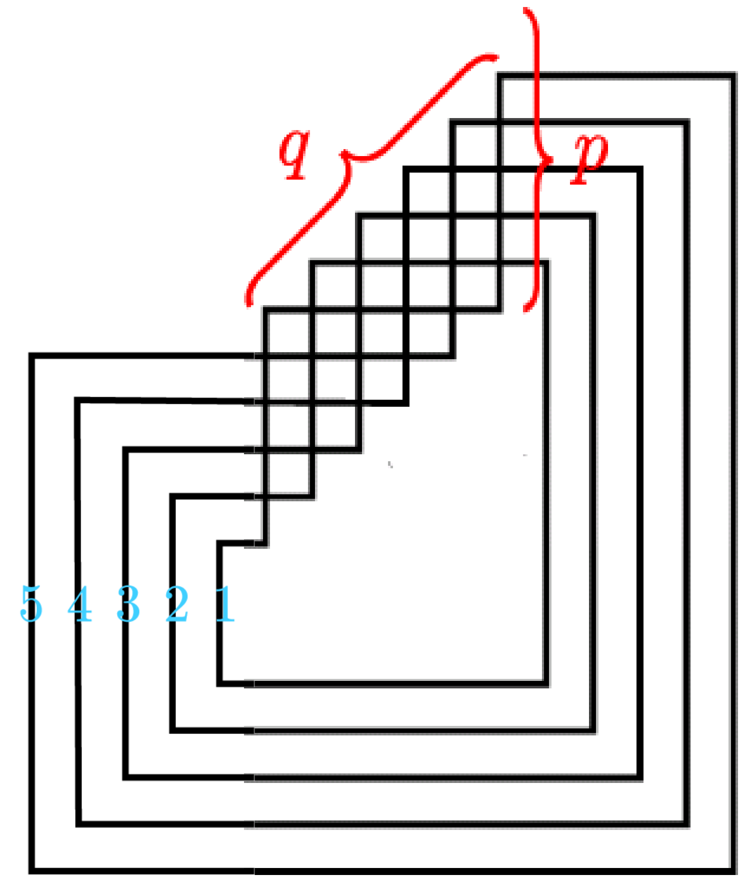
Theorem (Manturov 2002)

Every link can be represented as a closed quasitoric braid.

L : link

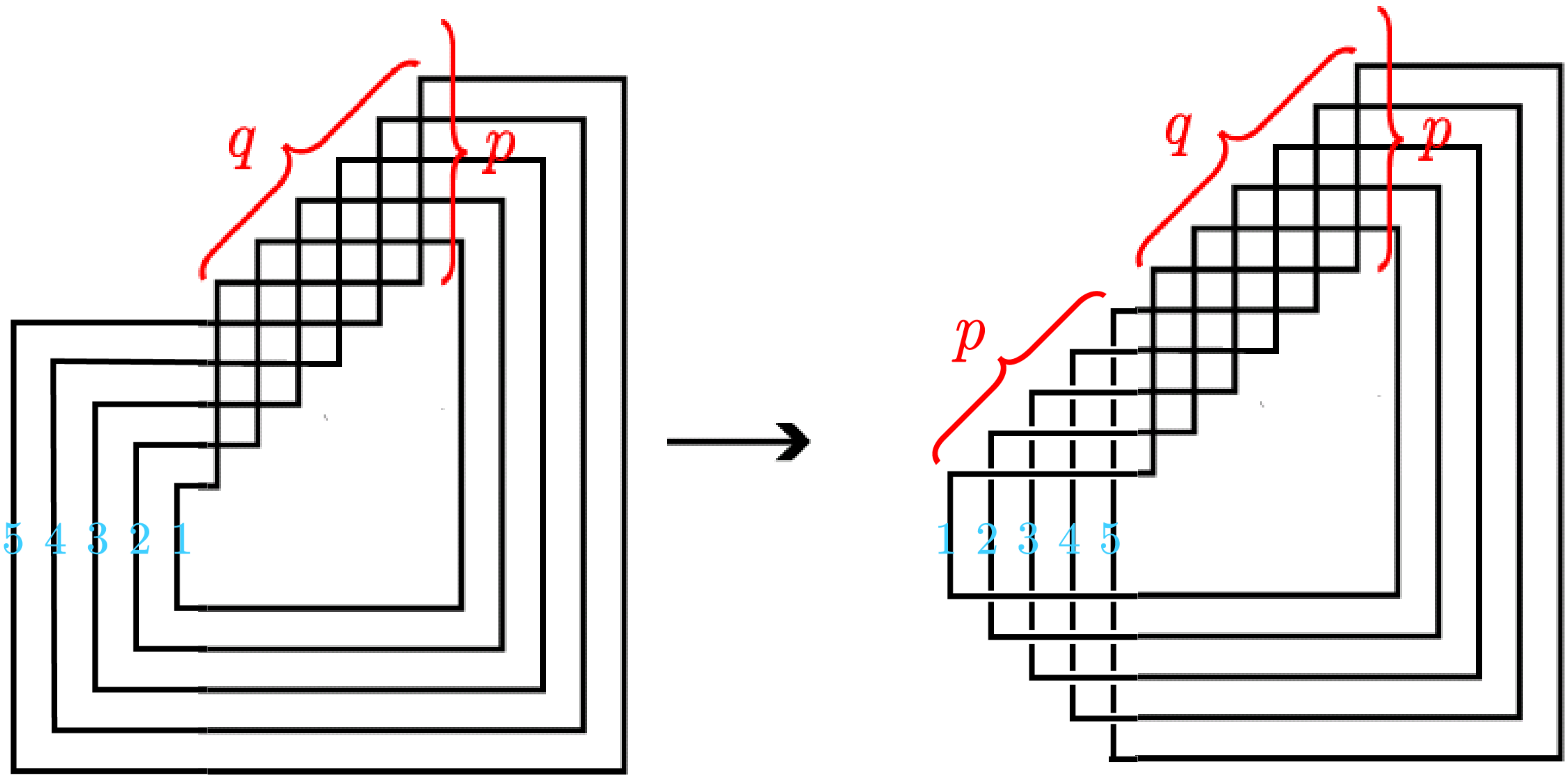


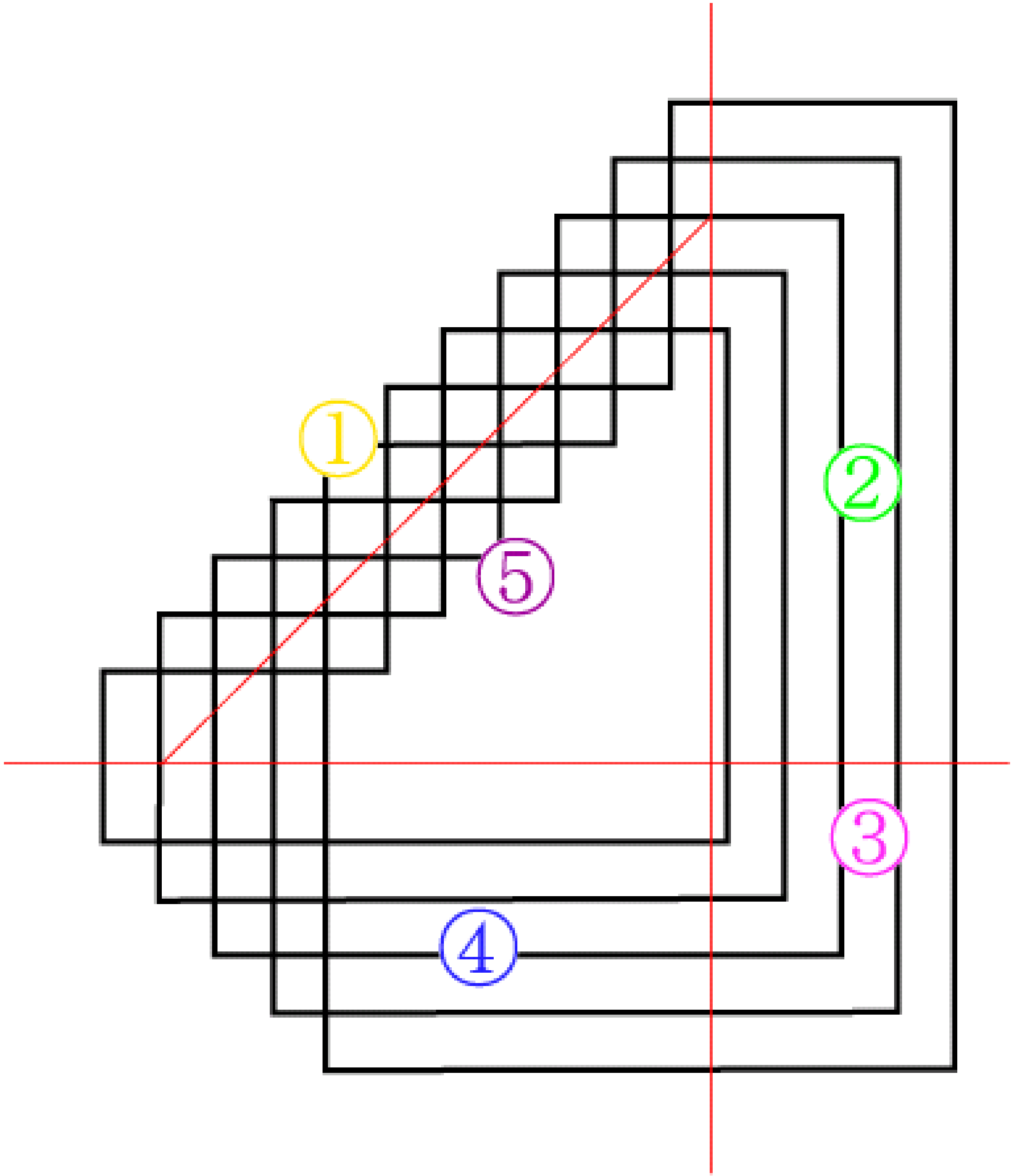
rectangularize



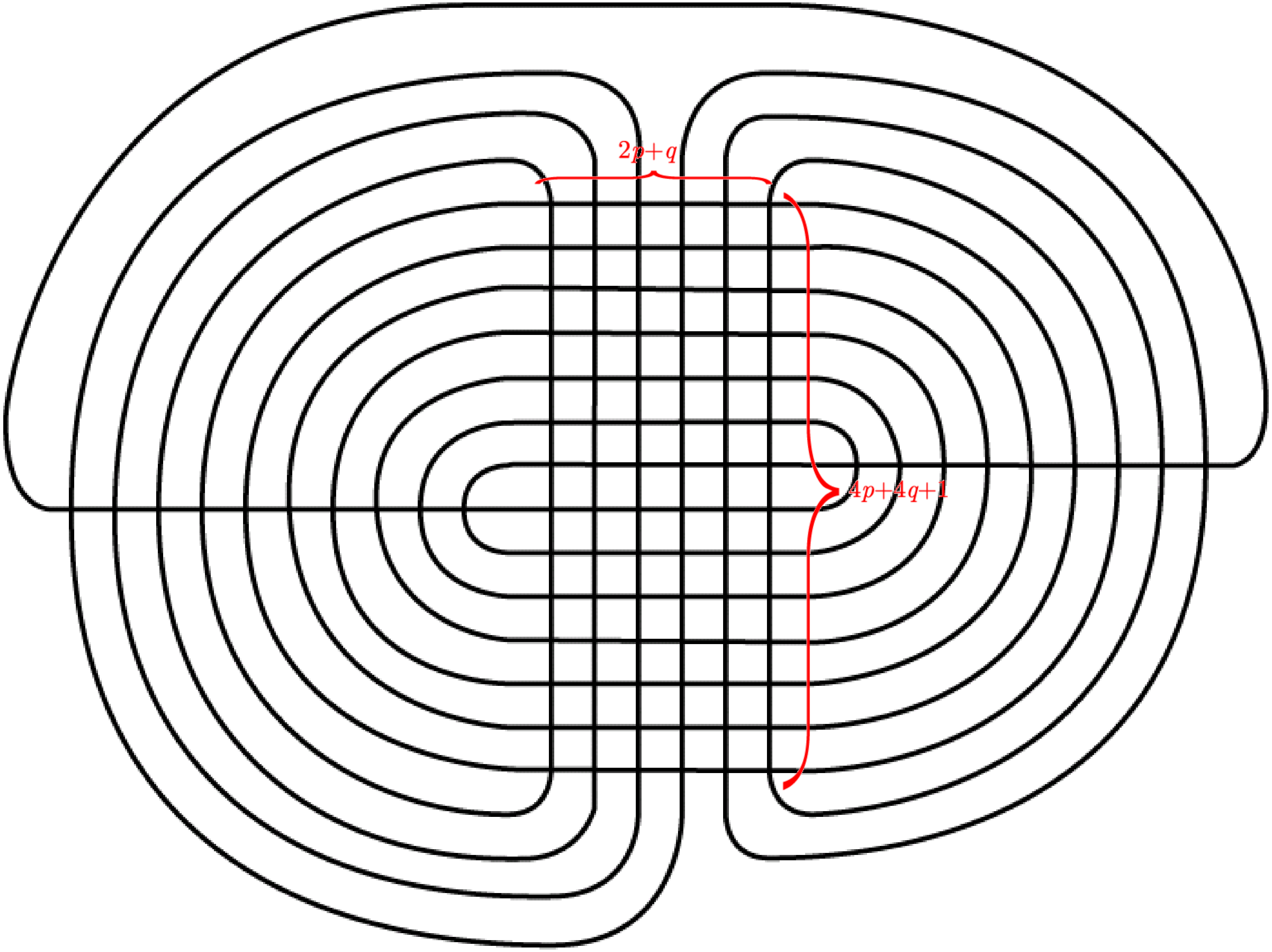
$(p = 5,\ q = 6)$

diagram of L represented
as a closed quasitoric braid

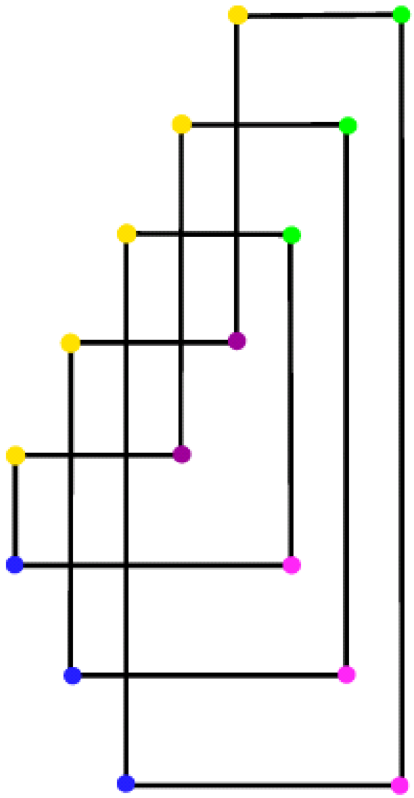

q
p
5 4 3 2 1
q
p
p
1 2 3 4 5

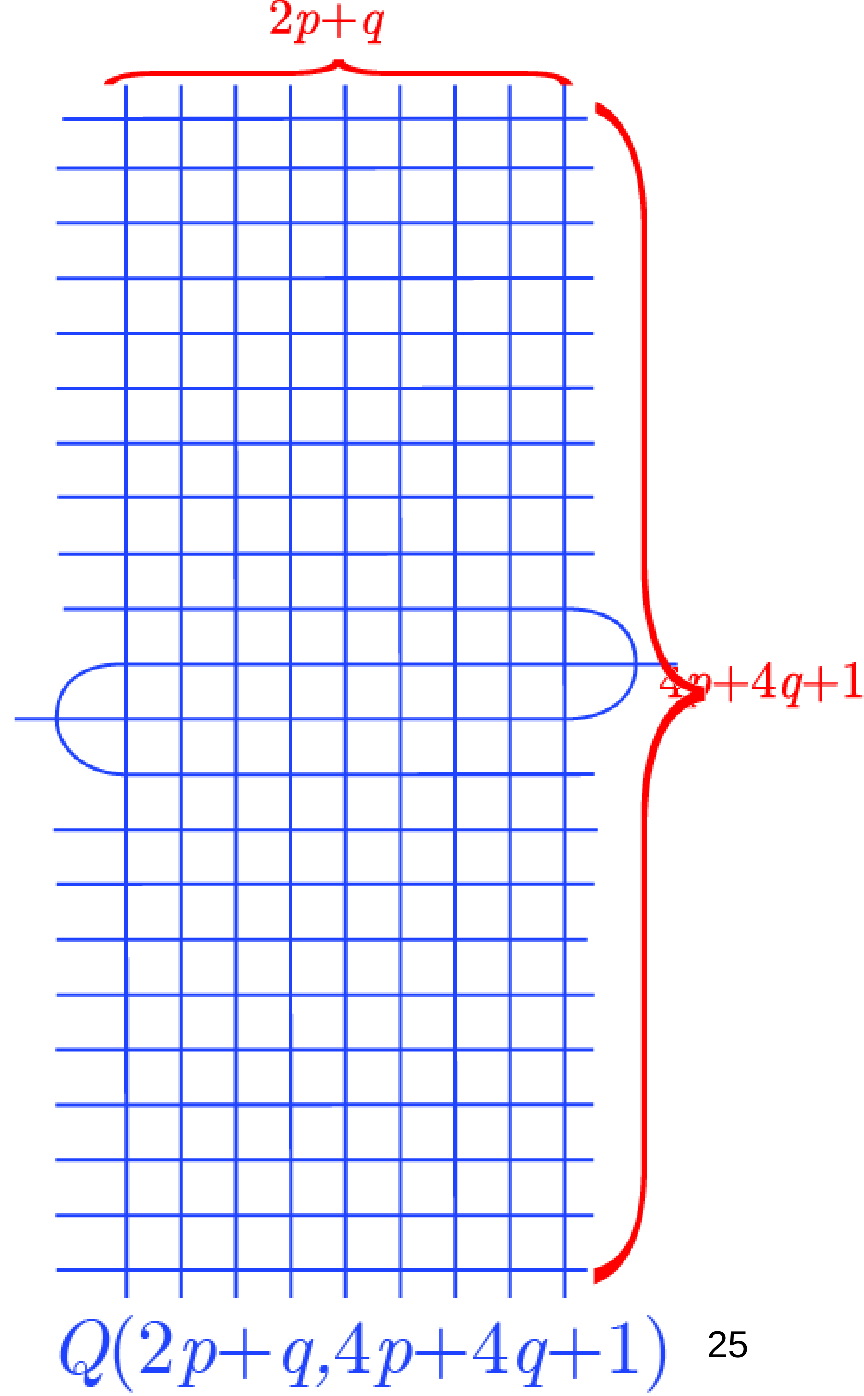


D' : diagram of L obtaided from
a closed quasitoric braid of type (p, q)

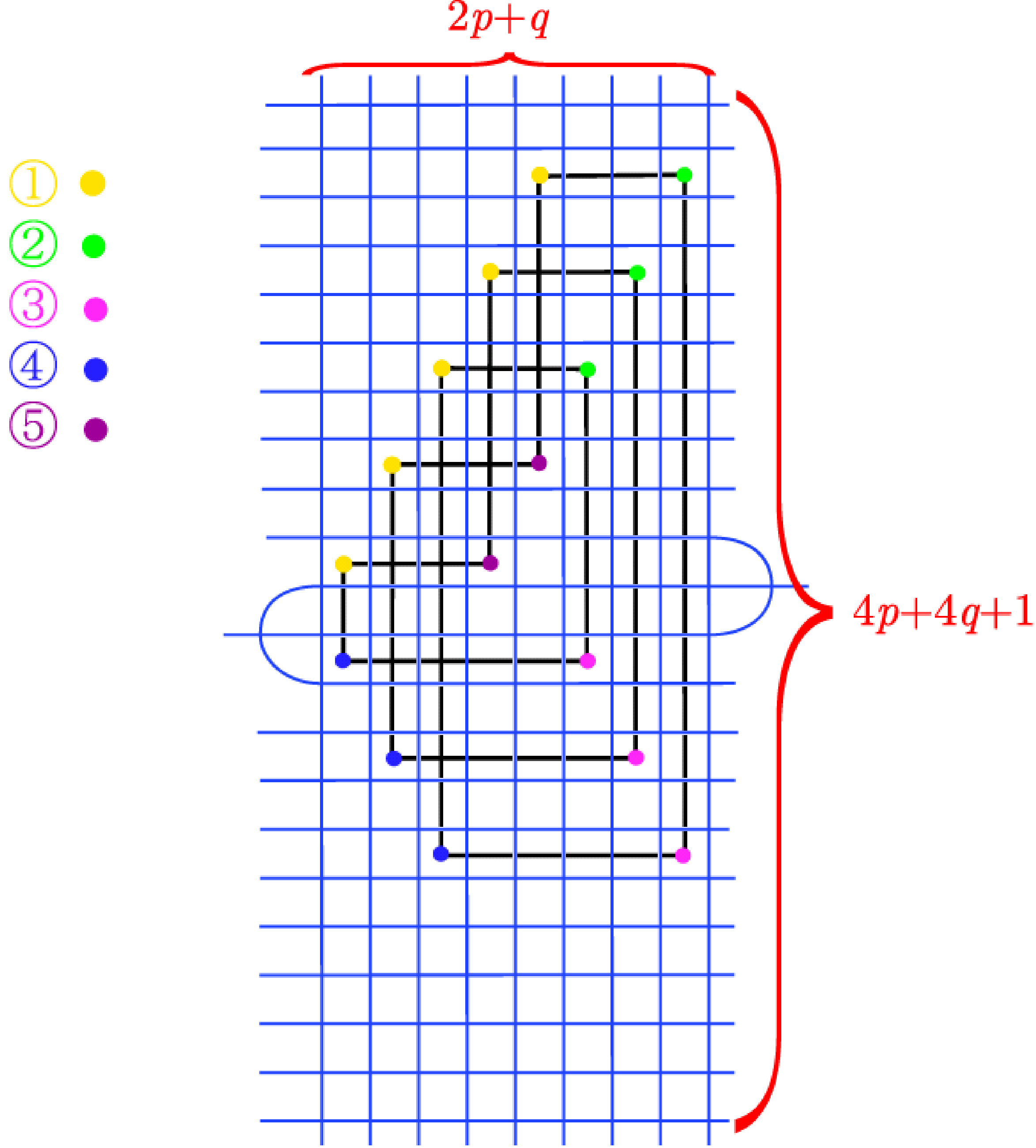


$Q(2p+q, 4p+4q+1)$

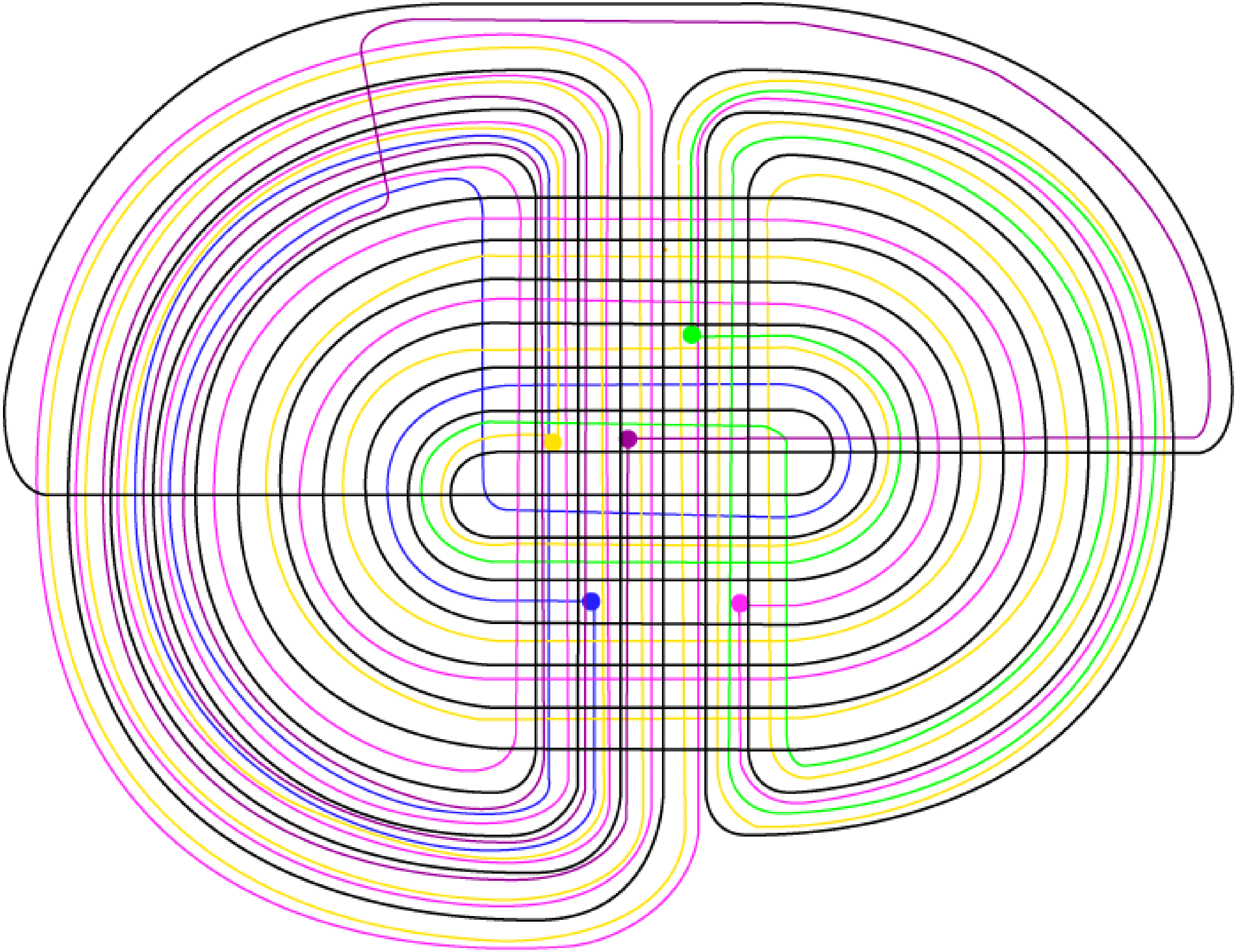




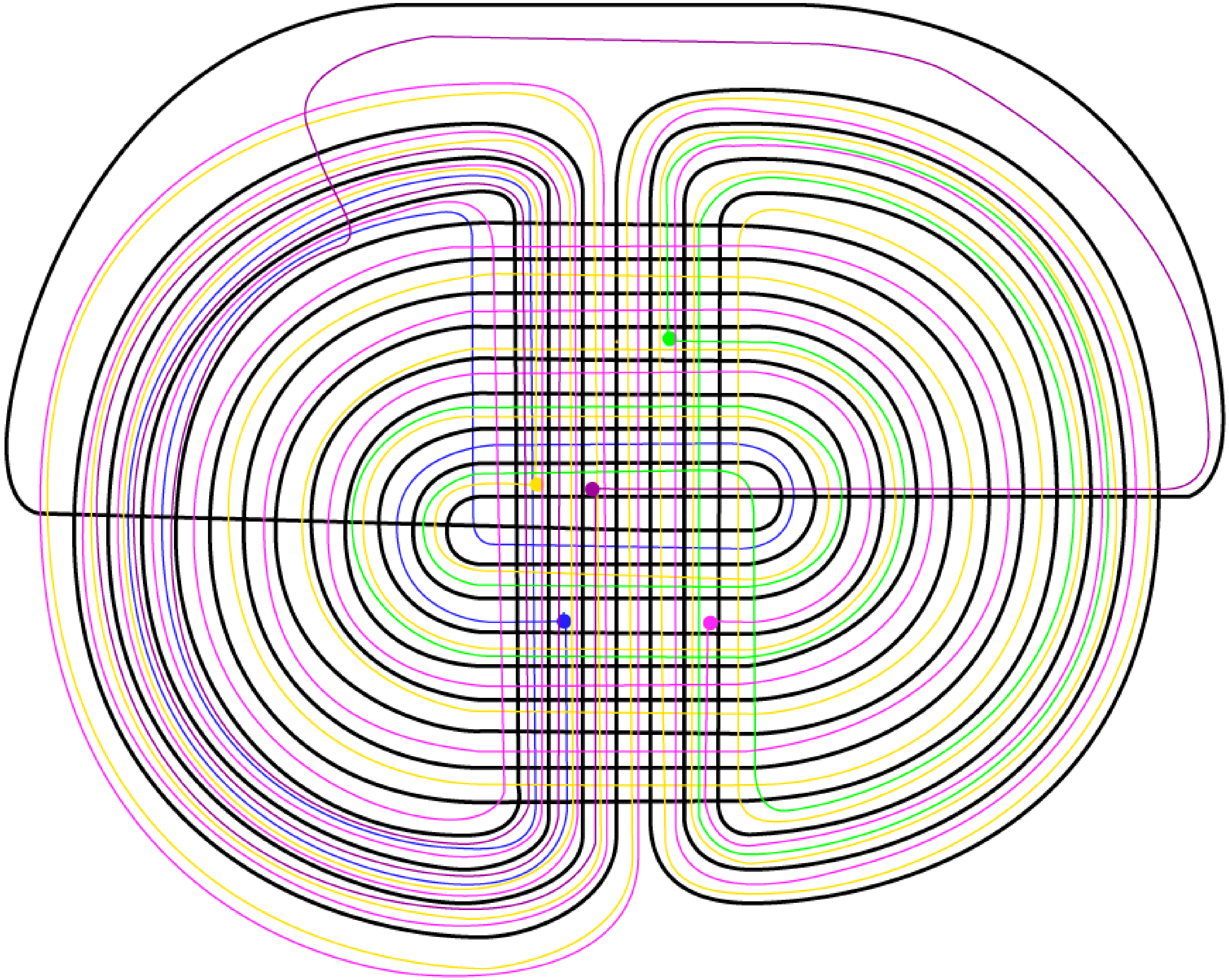
D': diagram of L obtaided from
a closed quasitoric braid of type (p, q)



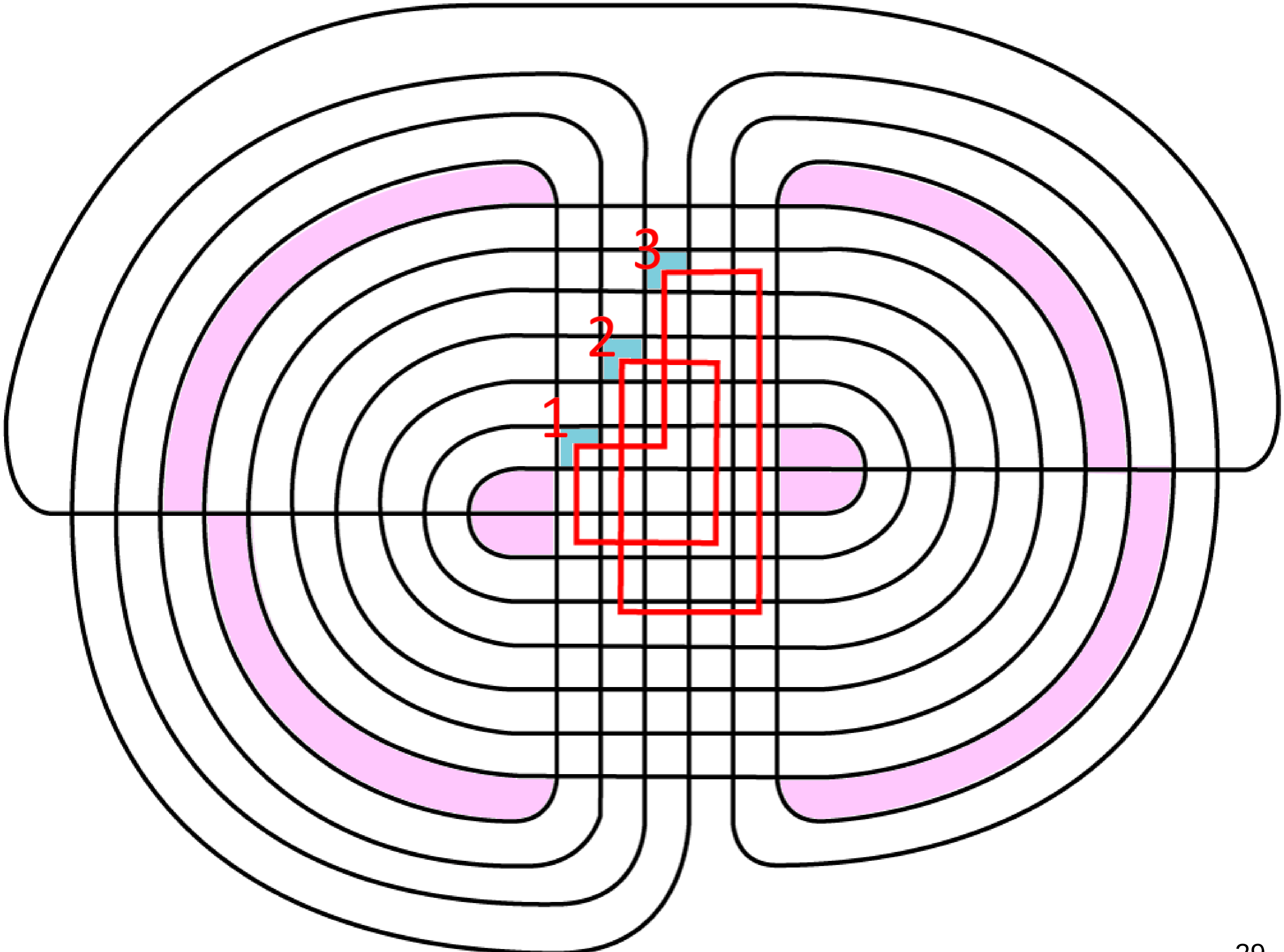
(3,4,5)-diagram of $L \bigsqcup T$

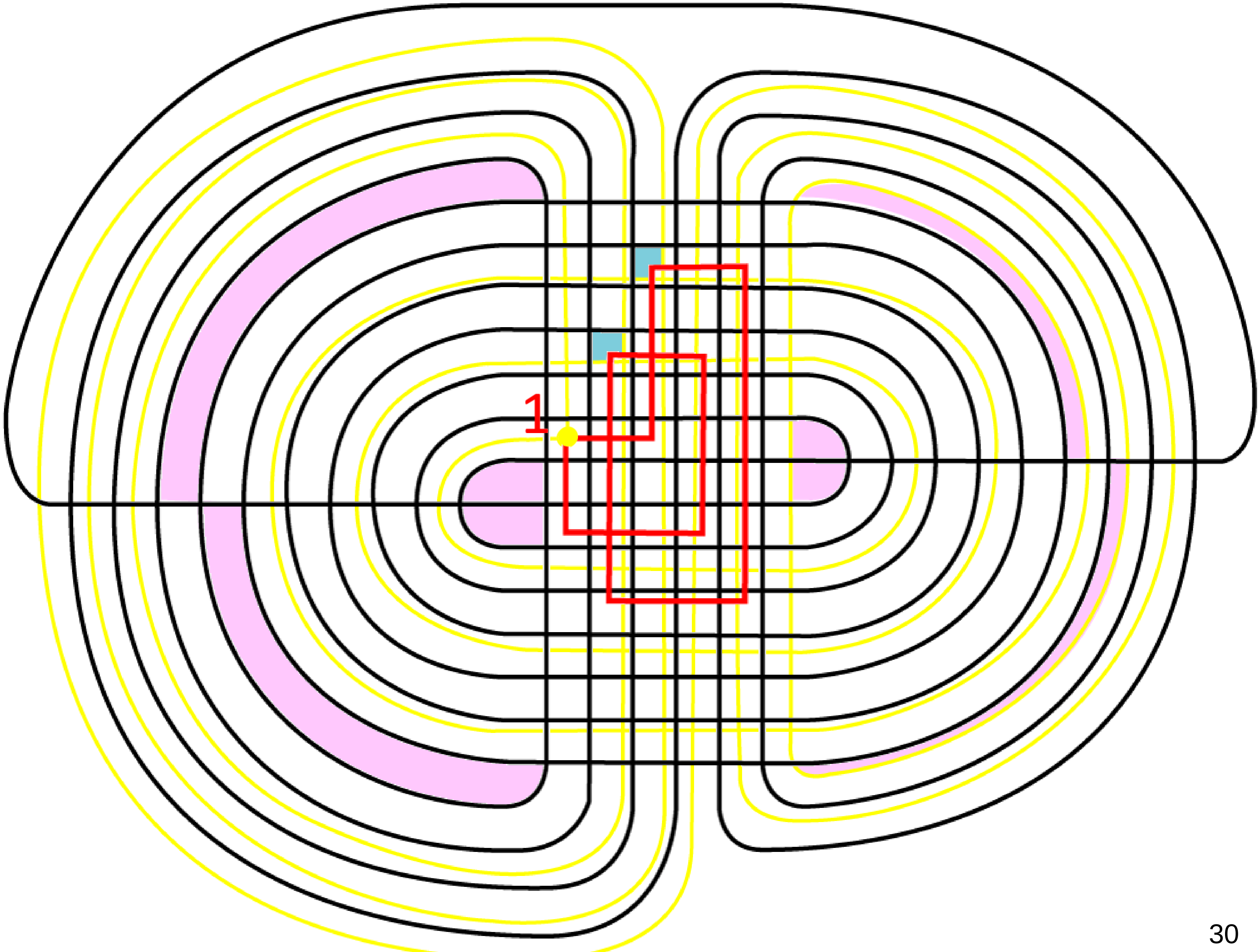


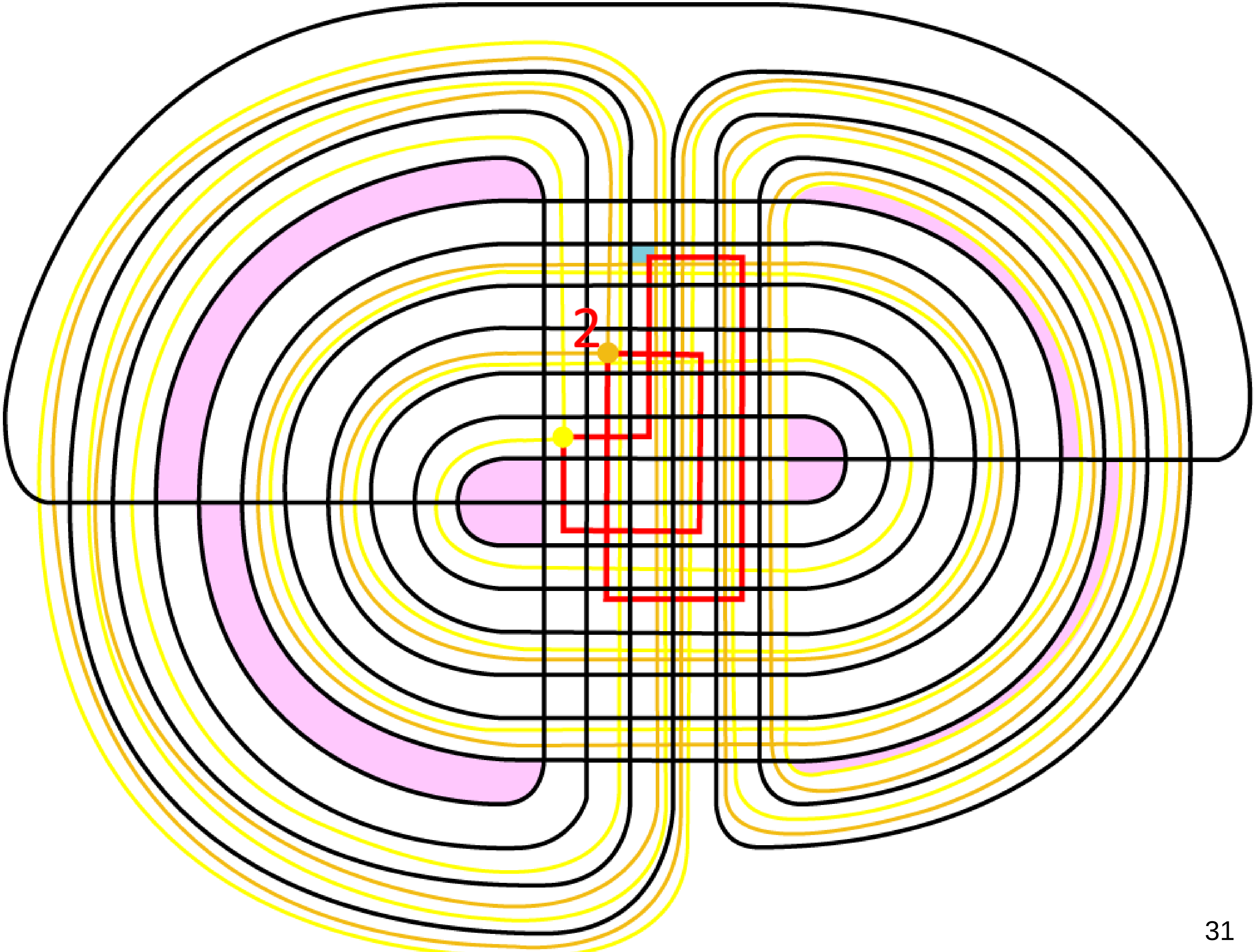
$Q(5,13)$

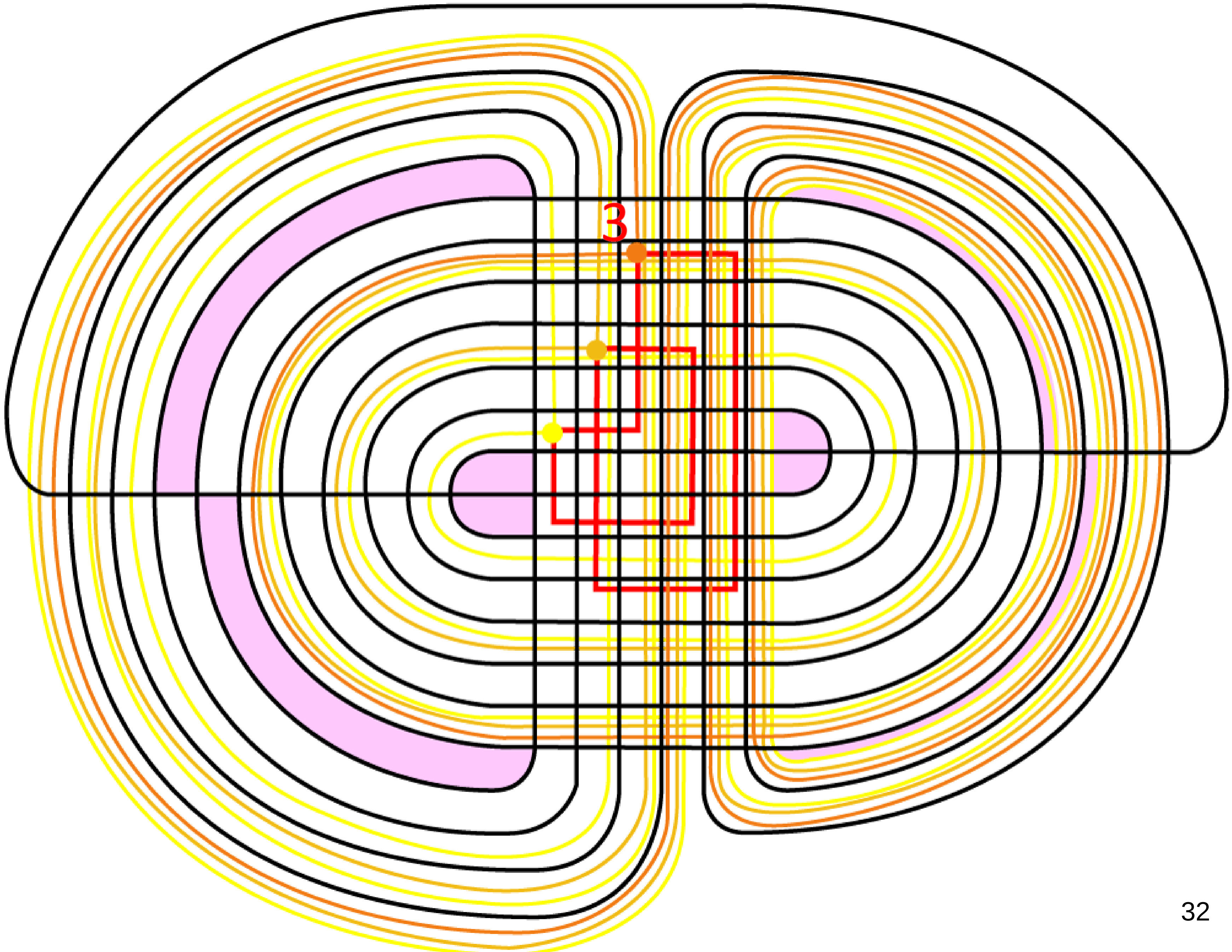


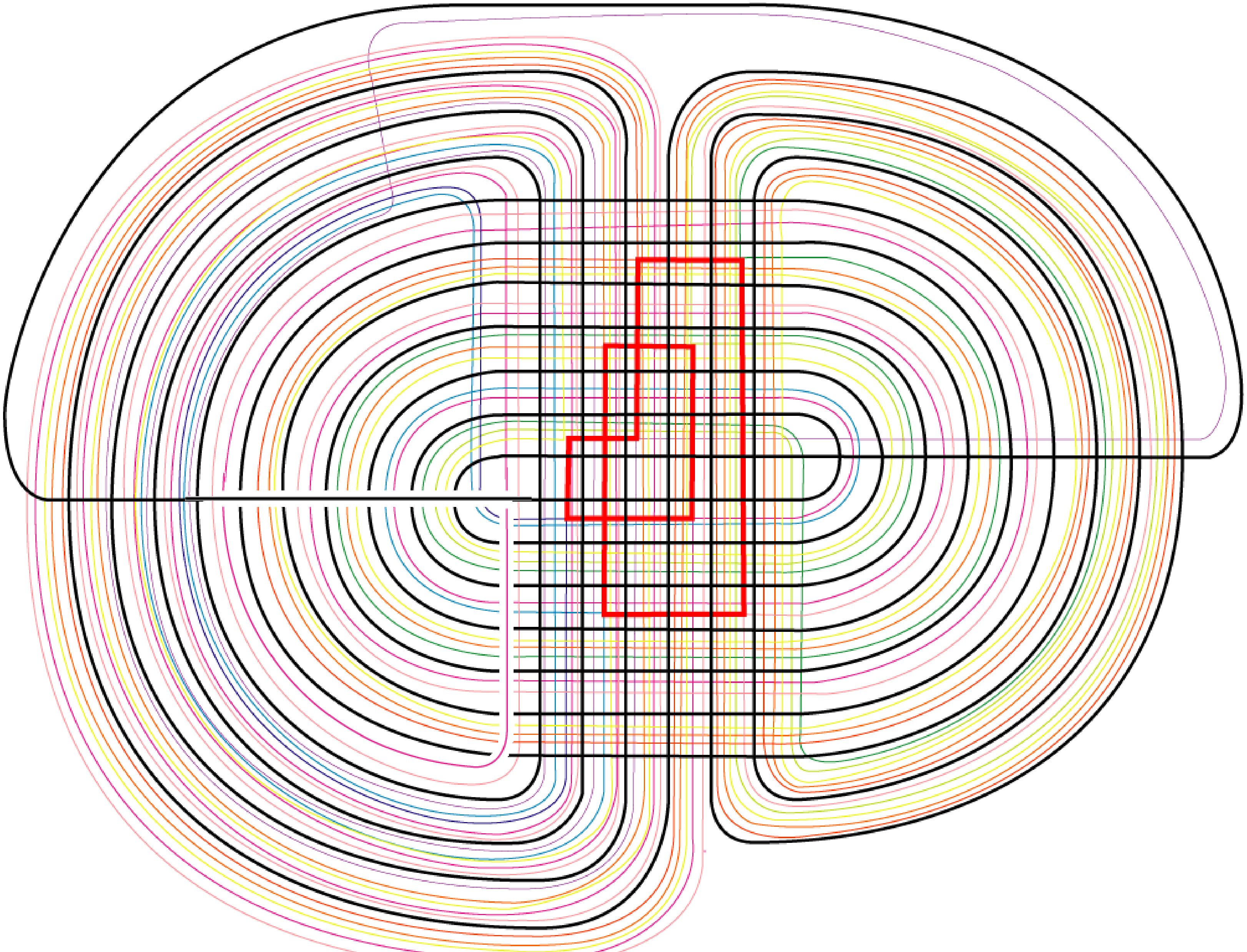
$Q(6,17)$


1
2
3

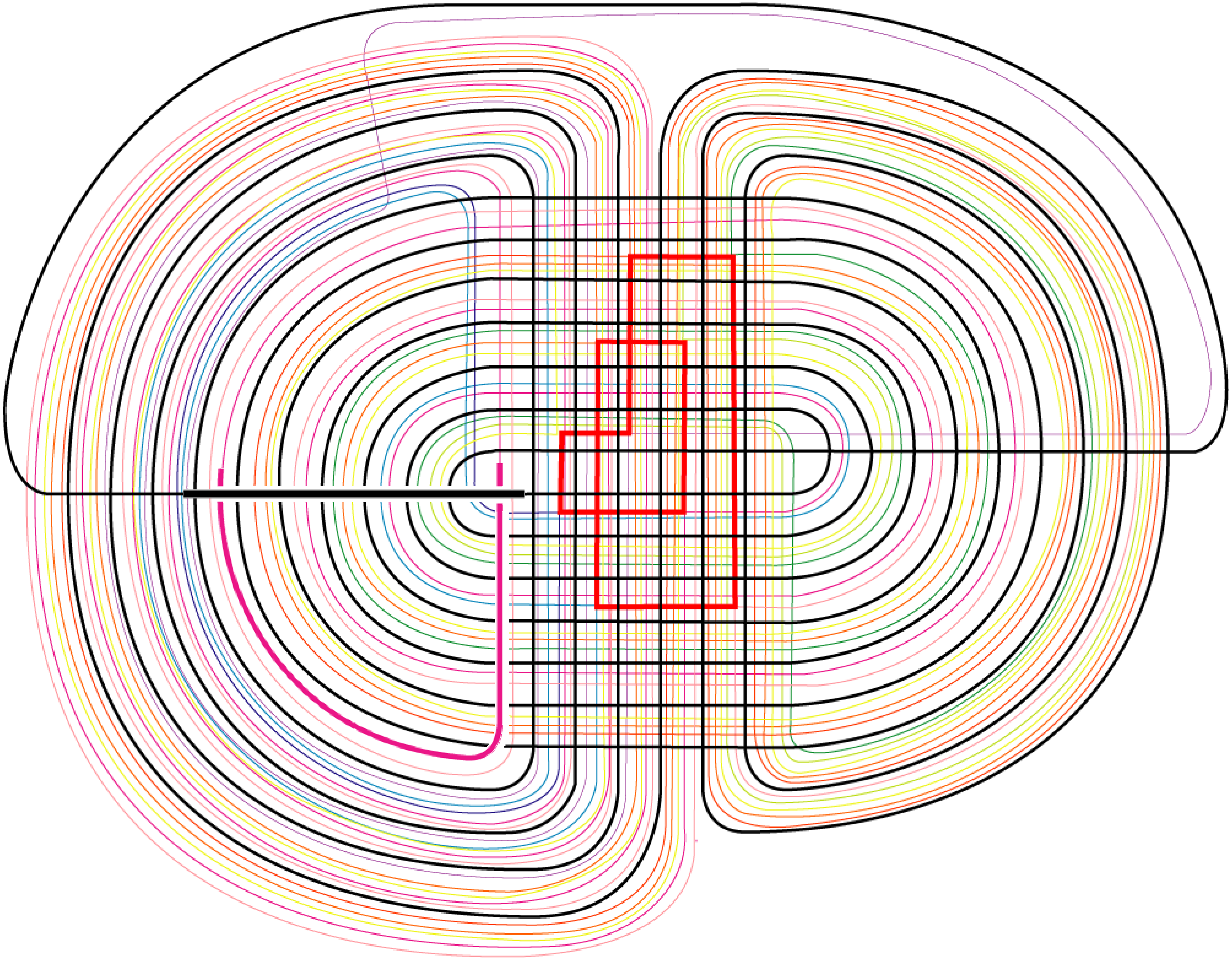

1

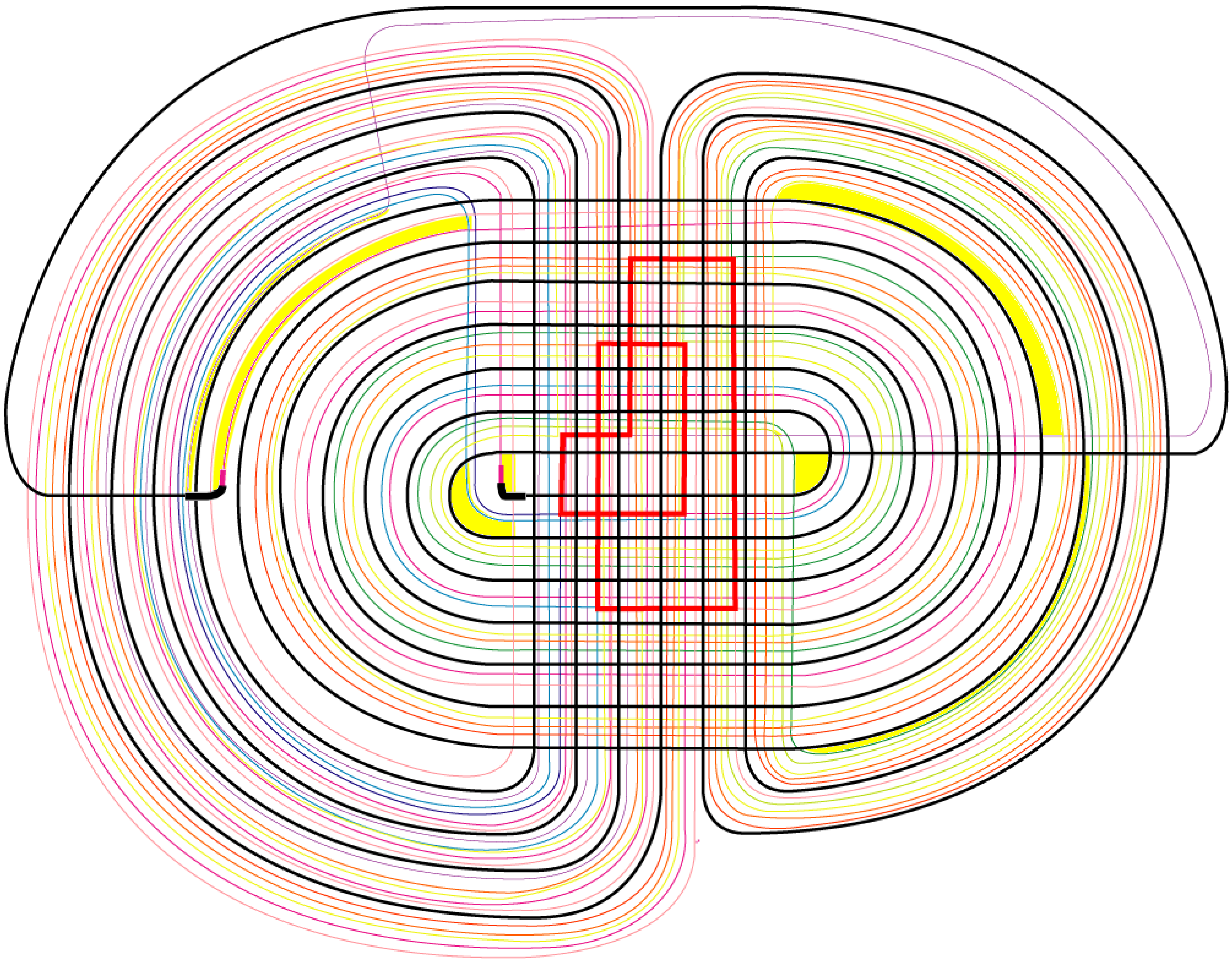

2


3



(3,4)-diagram of $L \bigsqcup T$





(3,4)-diagram of $L \sharp T$

Theorem (A-M-S)

(1) $\exists$ infinitely many knots which have a $(2,n)$-diagram.

$(n \not\equiv 0 \pmod 2)$

(2) $\exists$ infinitely many knots which have a $(3,n)$-diagram.

$(n \not\equiv 0 \pmod 3)$

Open problems

- Does every link have a $(2,4,n)$-diagram for some $n \geq 6$?
- Does every link have a S-diagram for some $S = (a_1, a_2, a_3)$ which does not contain 4?