Quantization of the crossing number of a knot diagram and quantities of the crossing points (joint work with Akio Kawauchi)

Ayaka Shimizu (OCAMI)

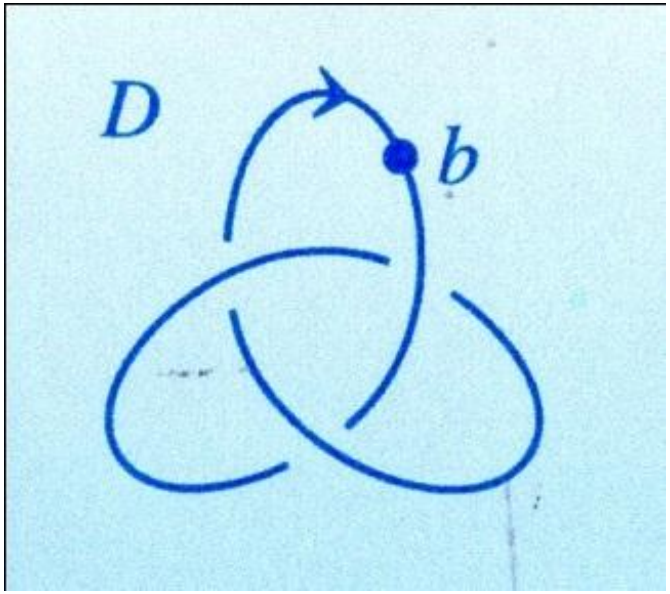
Contents

§ 1. Warping degree

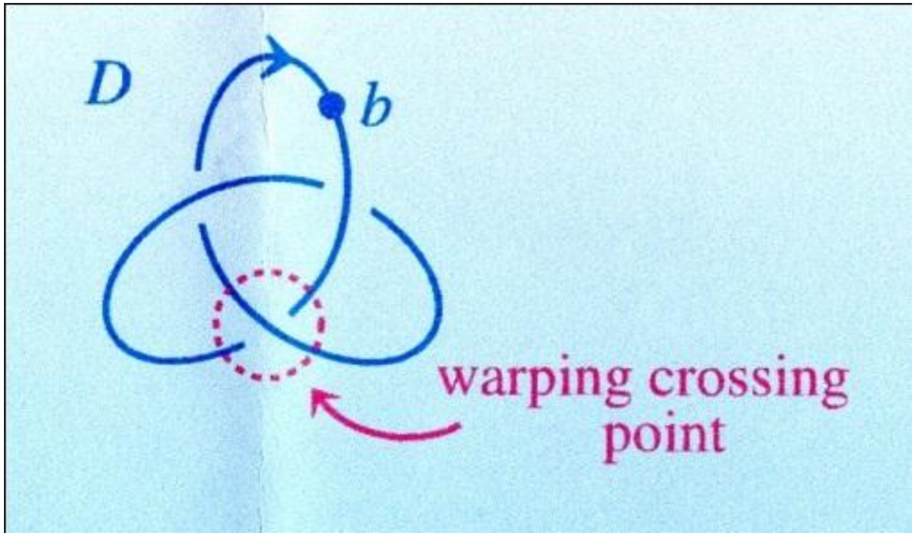
- Warping crossing point
- Warping degree

D: an oriented knot diagram on S^2
b: a base point of D

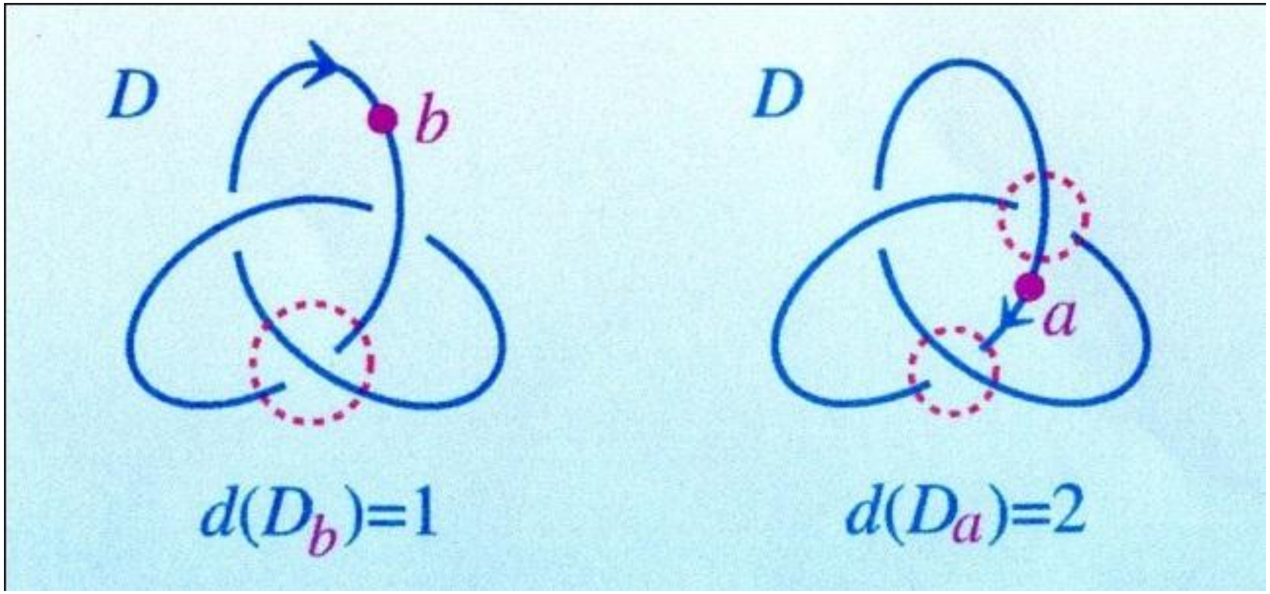
} D_b



A crossing point *p* of *D* is a warping crossing point of D_b if we come to *p* as an under-crossing first when we go along *D* by starting from *b*.



The warping degree $d(D_b)$ of D_b is the number of warping crossing points of D_b.

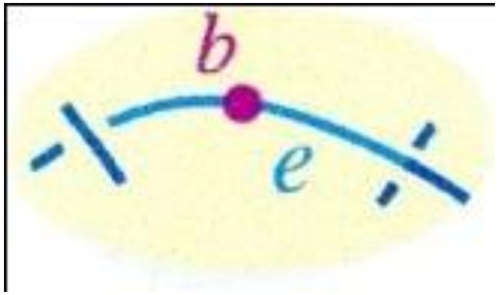


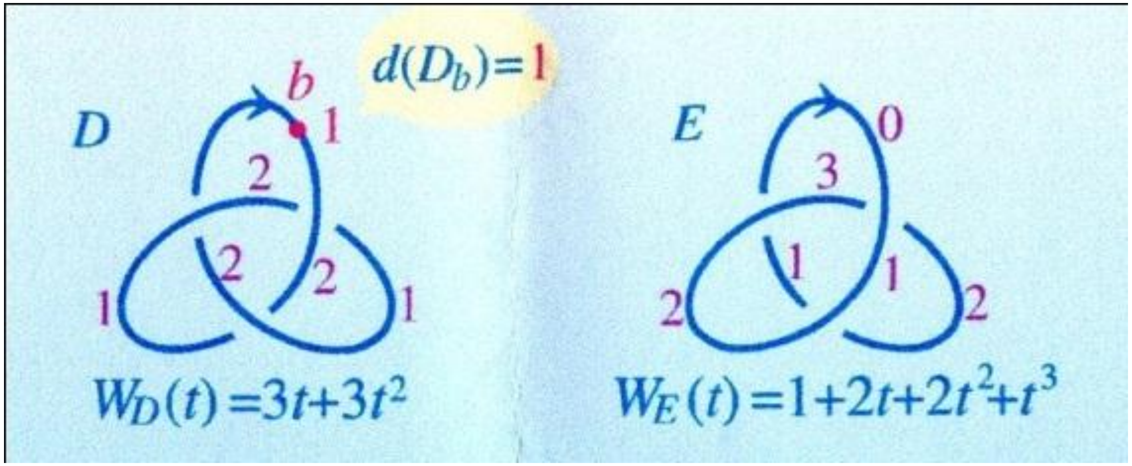
§ 2. Warping crossing polynomial

- Warping polynomial $W_D(t)$
- Warping crossing polynomial $X_D(t)$

The warping polynomial of *D*

$$W_D(t) = \sum_{e\text{: an edge of } D} t^{d(e)} \quad (d(e)=d(D_b),\ b\in e)$$



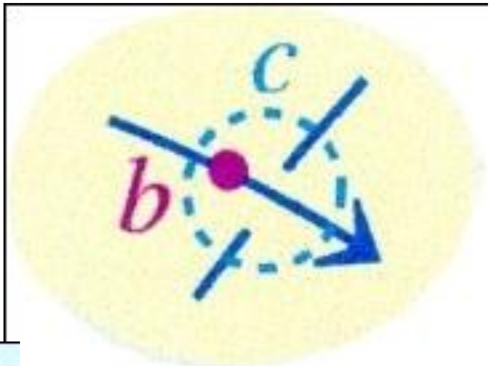


Reference: The warping polynomial of a knot diagram, A. Shimizu, arXiv:1109.5898

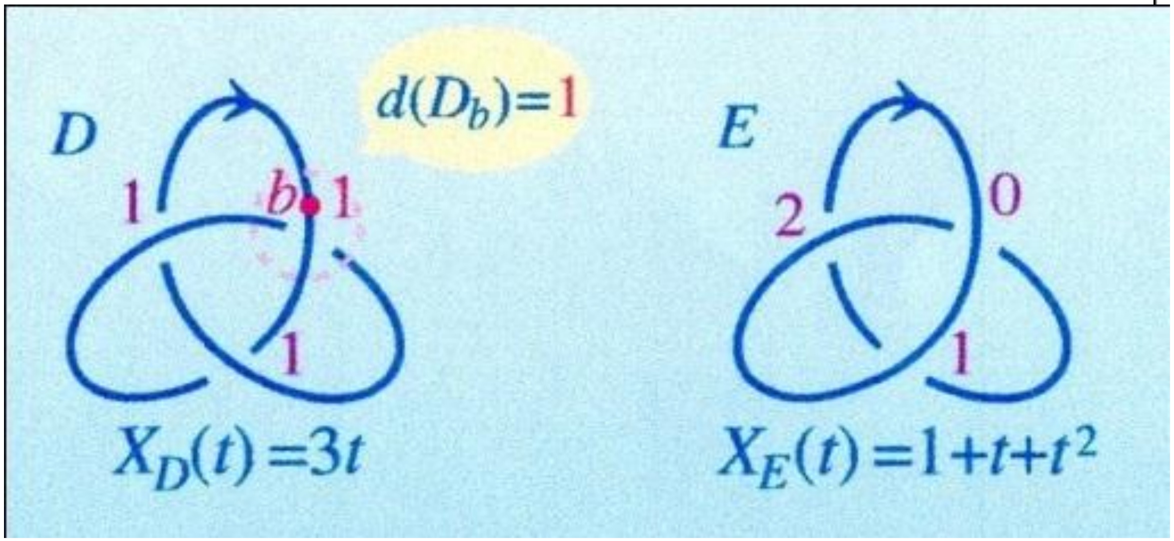
The warping crossing polynomial of D

$$X_D(t) = \sum t^{d(c)} \quad (d(c)=d(D_b),\ b \in e)$$

c: a crossing point of D



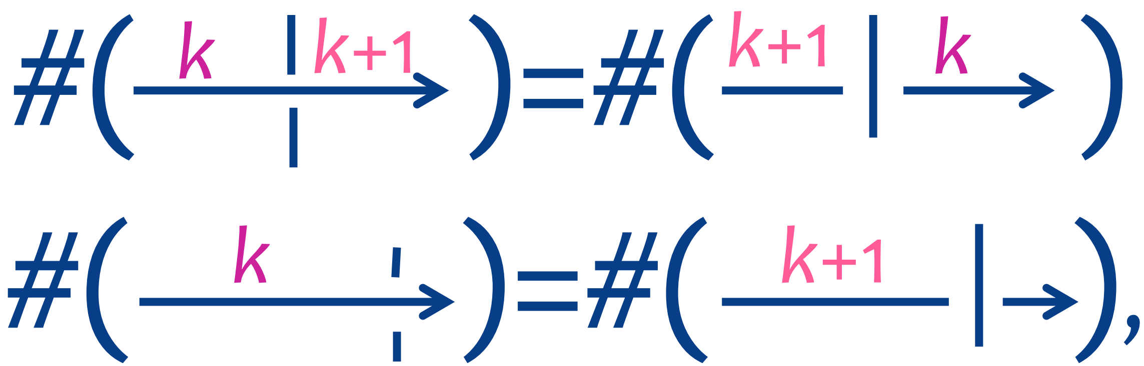
crossing weight



Proposition 1. $X_D(t) = \dfrac{W_D(t)}{1+t}$

Proof. Since

and



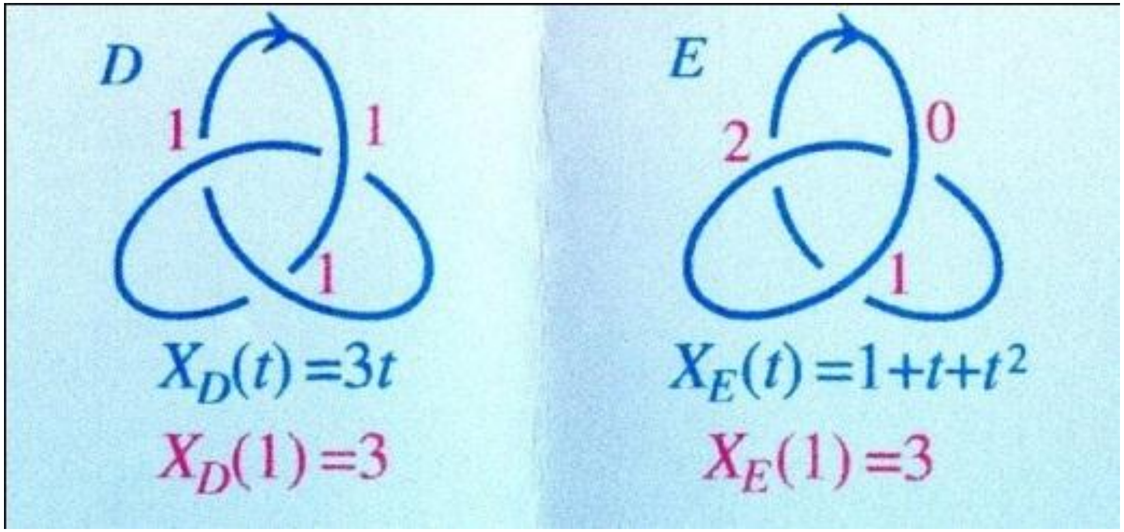
$$W_D(t) = \sum_{c} t^{d(c)} + \sum_{c} t^{d(c)+1} = (1+t)X_D(t).$$

□

§ 2. Warping crossing polynomial

Proposition 2.

Let D be an oriented knot diagram with the crossing number $c(D) \geqq 1$. We have $X_D(1)=c(D)$.



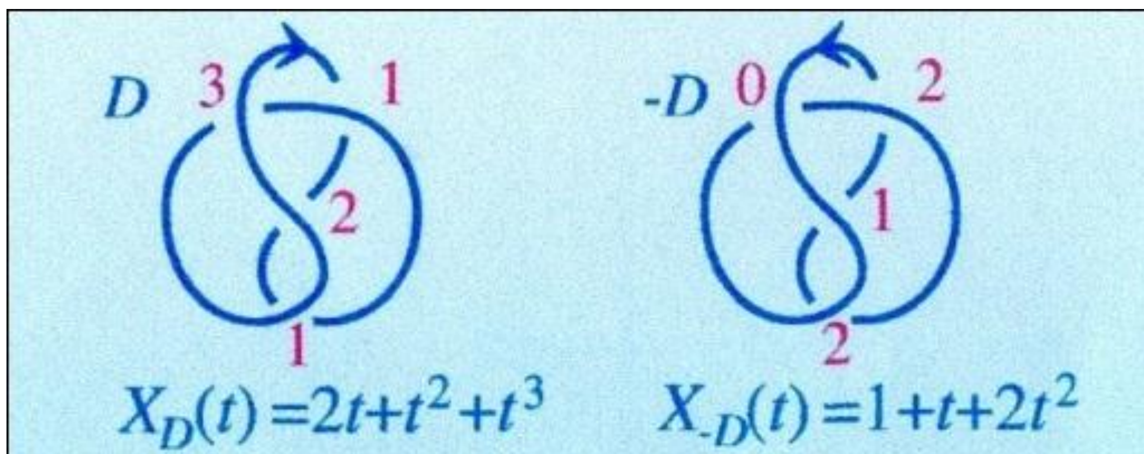
Let D be an oriented knot diagram with $c(D)=n \geqq 1$.

Proposition 3.

$-D$: D with orientation reversed

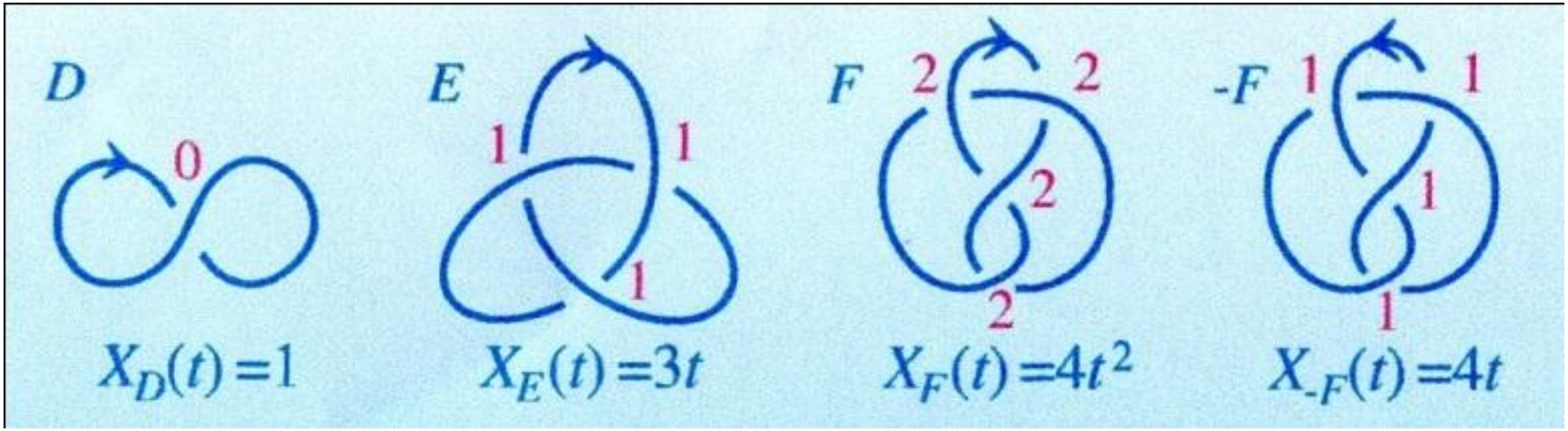
D^*: the mirror image of D

$$X_{-D}(t)=X_{D^*}(t)=t^{n-1}X_D(t^{-1}).$$



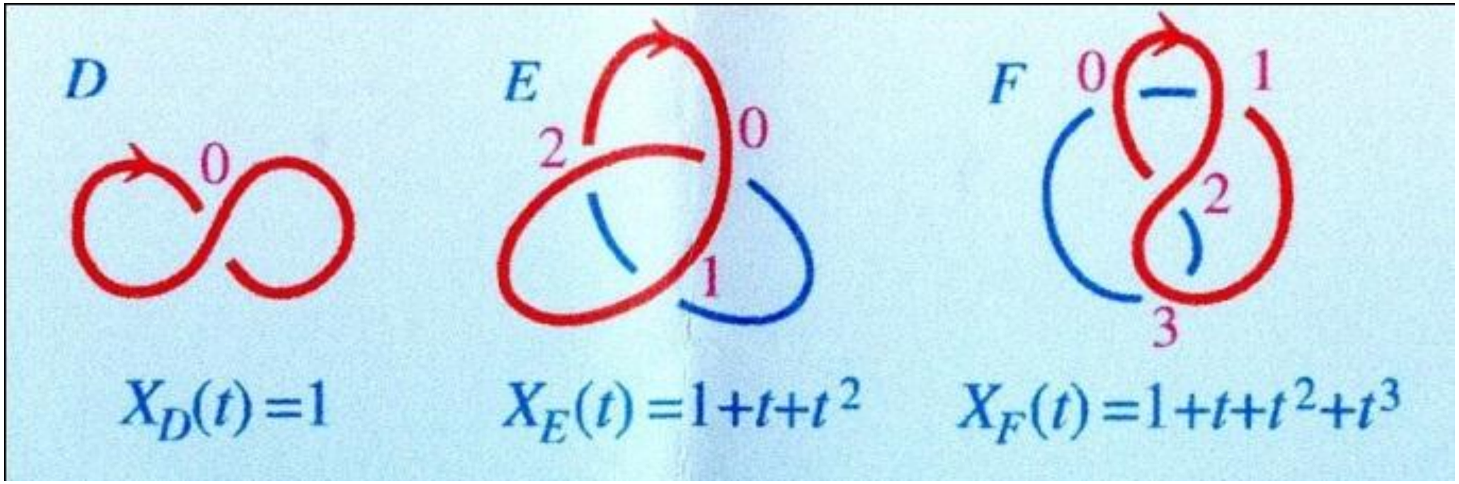
Proposition 4.

An oriented knot diagram D with $c(D)=n \geqq 1$ is alternating if and only if $X_D(t)=nt^d$ (d=0,1,2,…).



Proposition 5.

An oriented knot diagram D with $c(D)=n \geqq 1$ is a one-bridge diagram if and only if $X_D(t)=1+t+t^2+\ldots+t^{n-1}$.

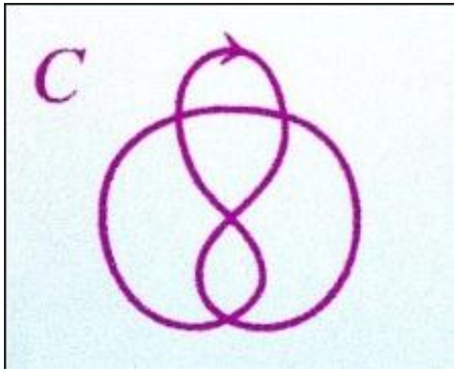


§ 3. State sum

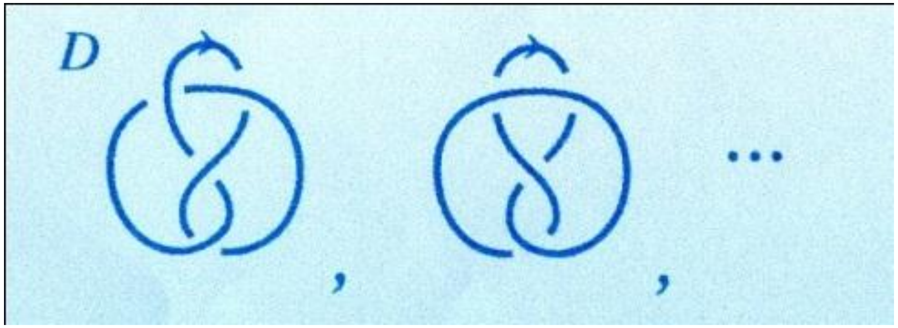
- State sum

- Crossing change

§ 3. State sum

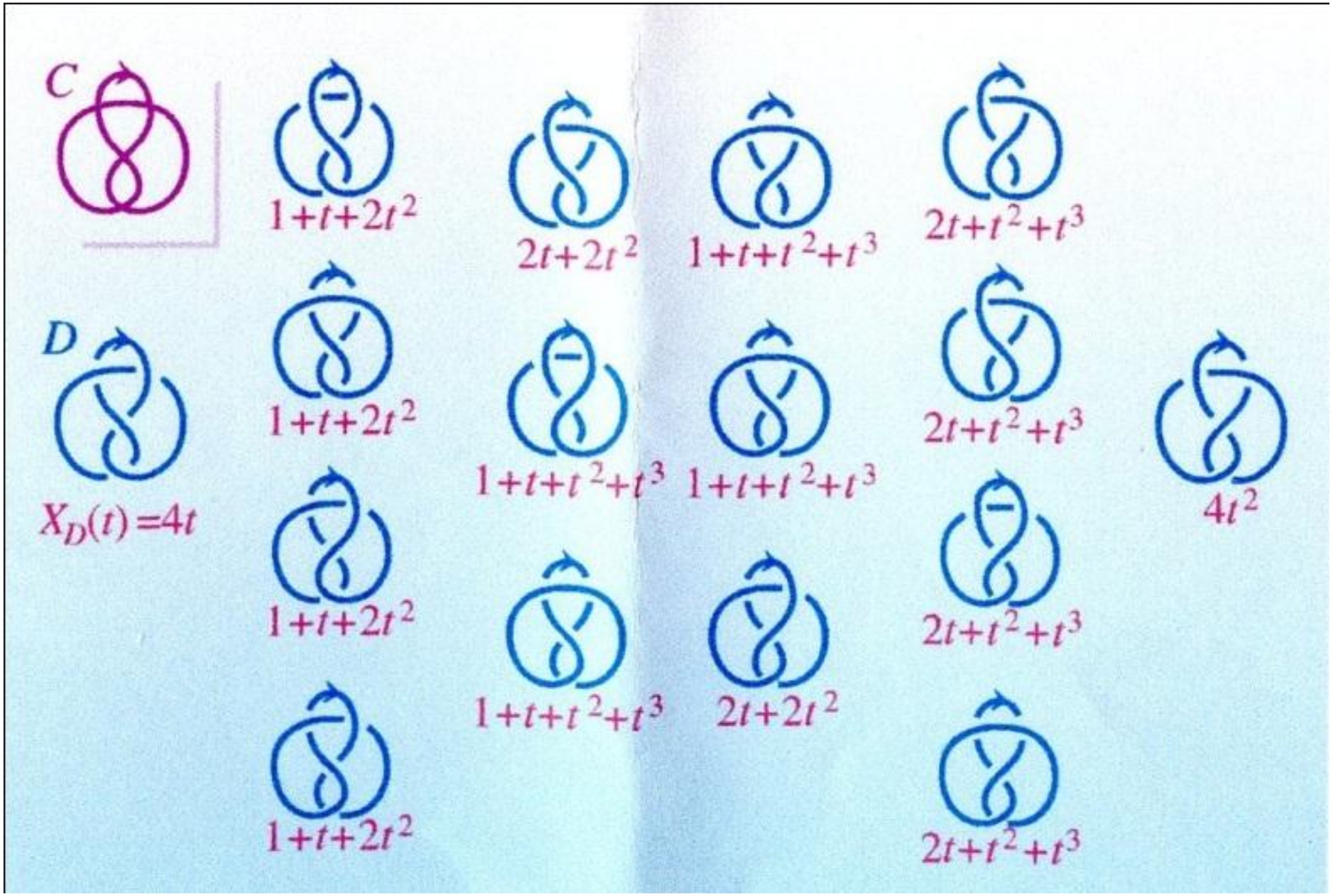


knot
projection



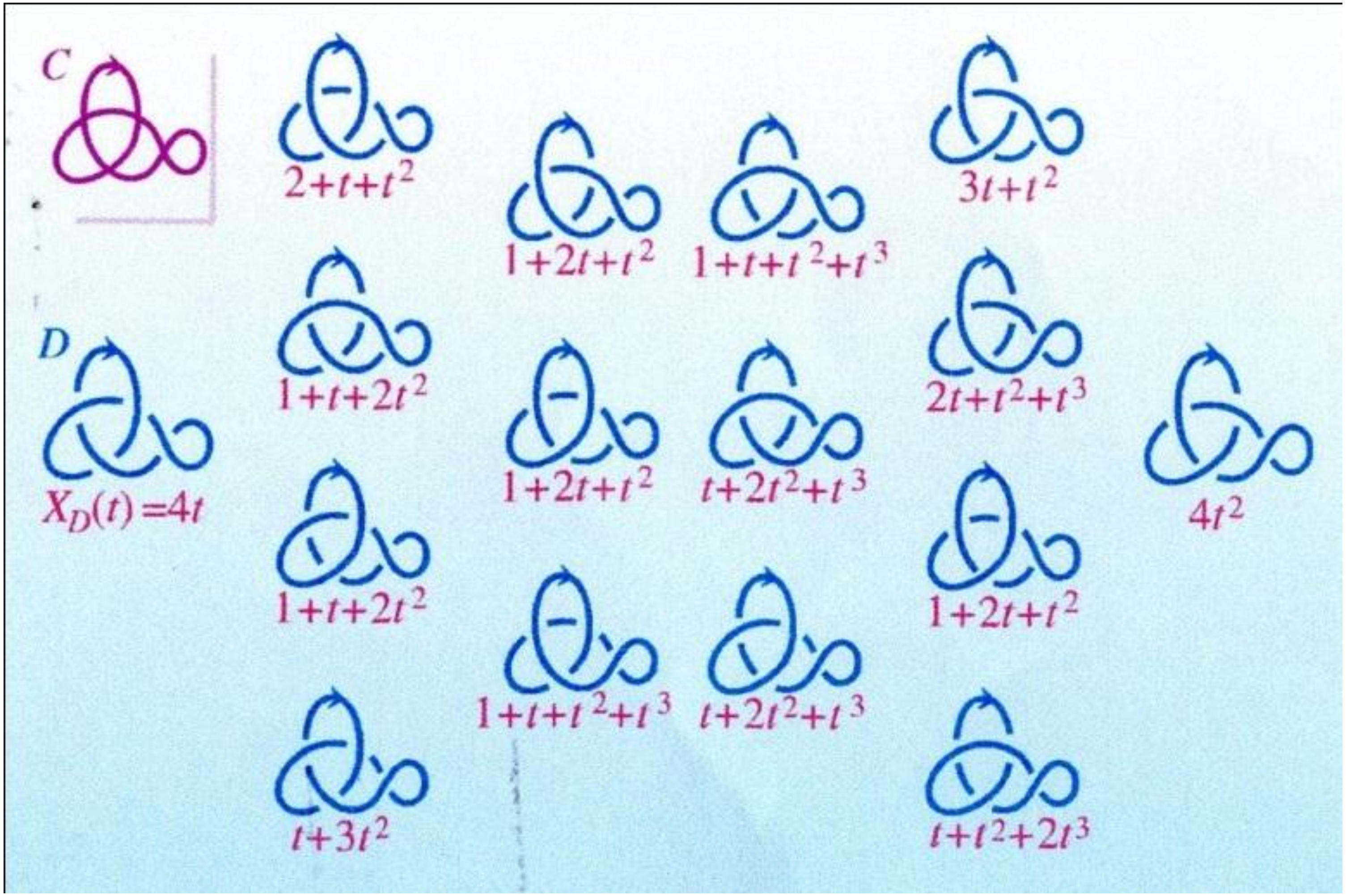
states
for C

§ 3. State sum



C
D
$X_D(t) = 4t$
$1+t+2t^2$
$2t+2t^2$
$1+t+t^2+t^3$
$2t+t^2+t^3$
$1+t+2t^2$
$1+t+t^2+t^3$
$1+t+t^2+t^3$
$2t+t^2+t^3$
$4t^2$
$1+t+2t^2$
$2t+t^2+t^3$
$1+t+t^2+t^3$
$2t+2t^2$
$1+t+2t^2$
$2t+t^2+t^3$

§ 3. State sum

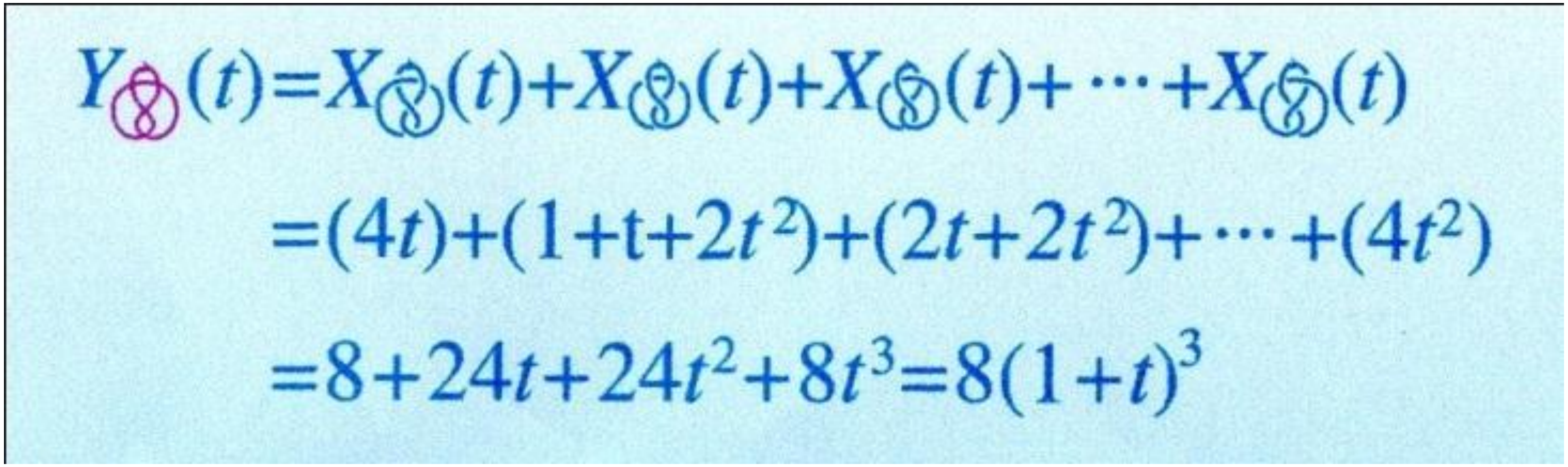


C
D
$X_D(t) = 4t$
$2+t+t^2$
$1+2t+t^2$
$1+t+t^2+t^3$
$3t+t^2$
$1+t+2t^2$
$1+2t+t^2$
$t+2t^2+t^3$
$2t+t^2+t^3$
$4t^2$
$1+t+2t^2$
$1+2t+t^2$
$1+t+t^2+t^3$
$t+2t^2+t^3$
$t+3t^2$
$t+t^2+2t^3$

The state sum

$$Y_C(t) = \sum_{\substack{D:\text{ a state}\\ \text{for } C}} X_D(t).$$

Example.



$$Y(t) = X(t) + X(t) + X(t) + \cdots + X(t)$$
$$= (4t) + (1 + t + 2t^2) + (2t + 2t^2) + \cdots + (4t^2)$$
$$= 8 + 24t + 24t^2 + 8t^3 = 8(1+t)^3$$

Theorem 6.

Let C be an oriented knot projection with n crossings ($n \geqq 1$). Then we have

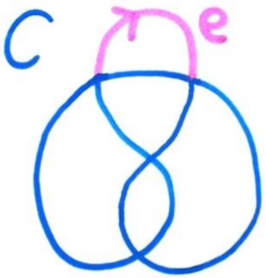
$$Y_C(t) = 2n(1+t)^{n-1}.$$

Remark. $Y_{-C}(t) = Y_C(t)$.

Lemma 7.

$$\sum_{D\text{: a state for } C} W_D(t) = 2n(1+t)^n.$$

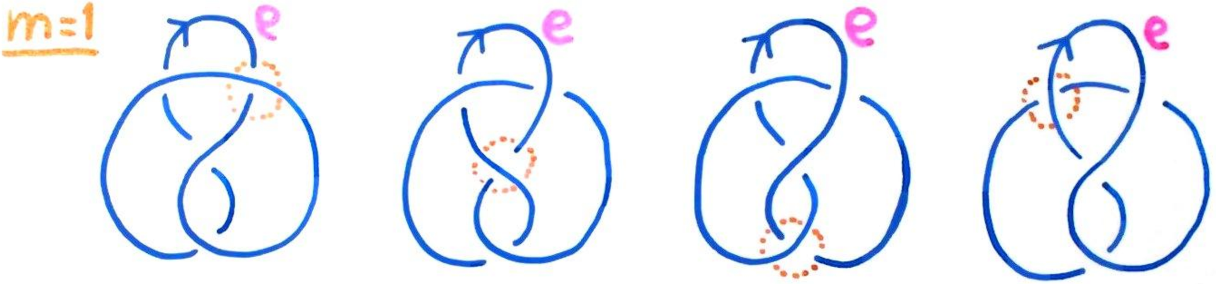
Proof of Lemma 7.



C: an oriented knot projection

e: an edge of C

We can replace all the double points with crossing points so that *e* has *m* warping crossing points (m=0,1,…,n) in $_nC_m$ ways.



Hence

$$\sum_{D:\text{ a state for } C} W_D(t) = 2n \times {}_nC_0 + 2n \times {}_nC_1 t + 2n \times {}_nC_2 t^2 + \ldots + 2n \times {}_nC_n t^n$$

$$= 2n(1+t)^n.$$

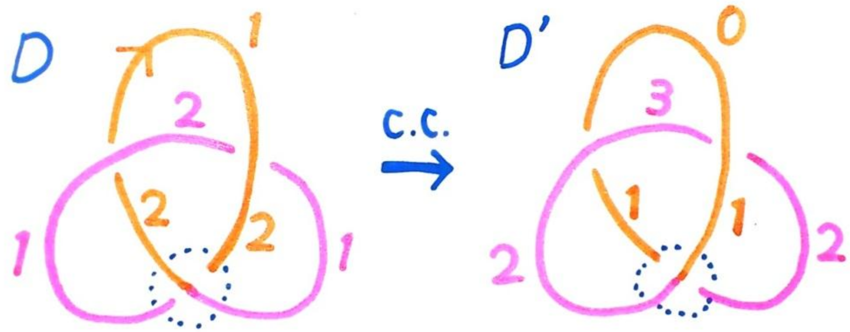
□

Proposition 8.

Let D be an oriented knot diagram, and let D' be the diagram obtained from D by a crossing change. Then,

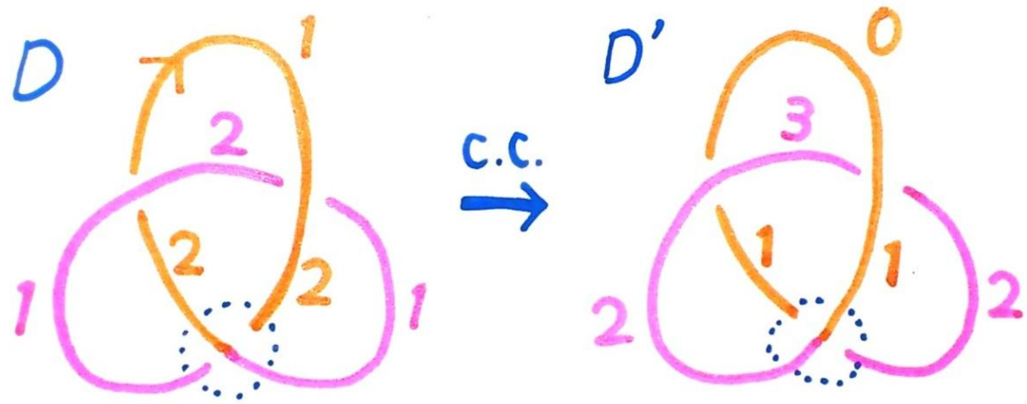
$$X_D(t) = \frac{A+B}{1+t}, \qquad X_{D'}(t) = \frac{tA+t^{-1}B}{1+t},$$

where A and B are polynomials.



Corollary 9.

$$\left| \operatorname{span} X_{D'}(t) - \operatorname{span} X_D(t) \right| \leqq 2.$$



span $X_D(t) = 0$ span $X_{D'}(t) = 2$

§ 4. Orientations of plane curves

The warping degree $d(D)$ of D is the minimal degree of $X_D(t)$.

Theorem 10(S. 2010).

$$d(D)+d(-D) \leqq c(D)+1$$

"=" $\Leftrightarrow$ D is alternating and $c(D) \geqq 1$.

Corollary 11.

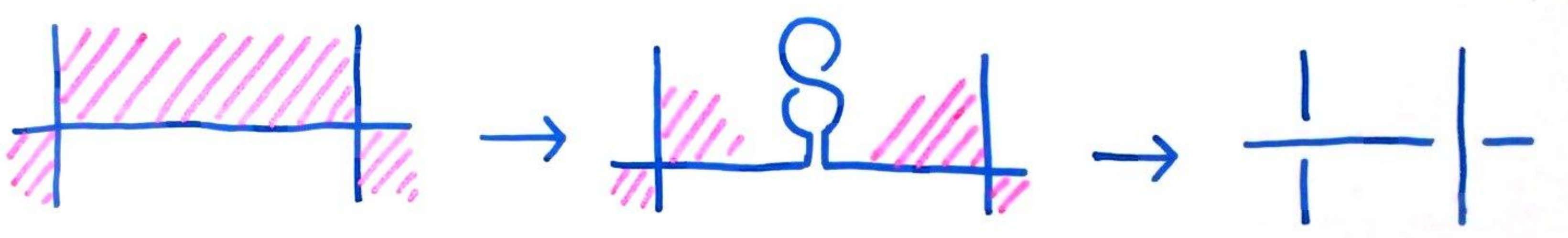
Let D be an oriented alternating knot diagram with non-zero even crossings. Then $d(D) \neq d(-D)$.

Theorem 12.

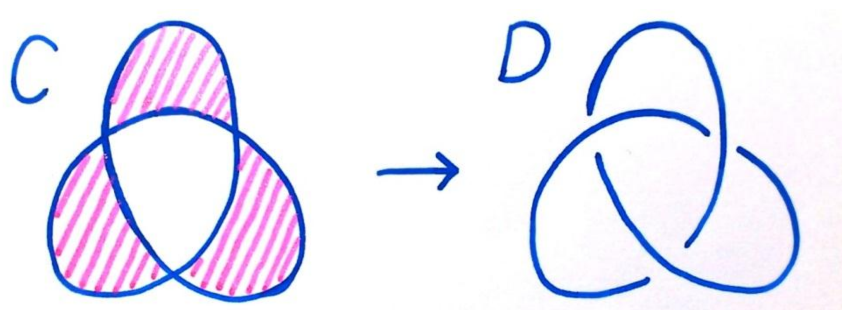
Every based closed transversely intersected plane curve (knot projection) C_b on $\boldsymbol{R}^2$ has a canonical orientation.

Proof of Theorem 12.

①We apply C the checkerboard coloring such that the outer region is colored white. Then C can be lifted to a unique alternating knot diagram D as follows:

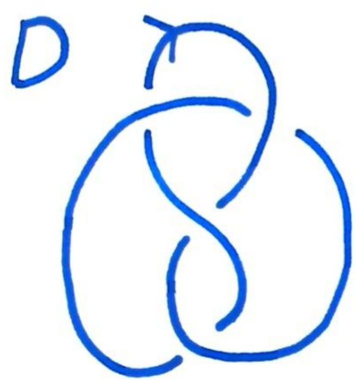


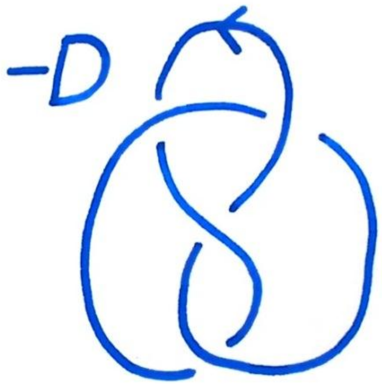
Example.



②・n:even

Give D the orientation such that $d(D)<d(-D)$.
Thus, C is oriented.



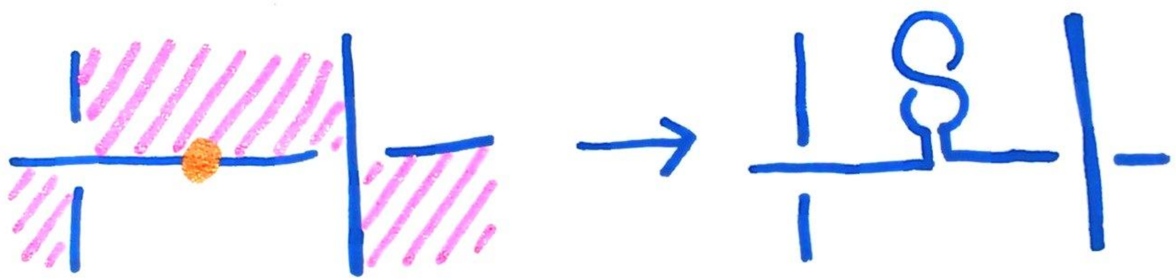


$X_D(t)=4t$
$d(D)=1$

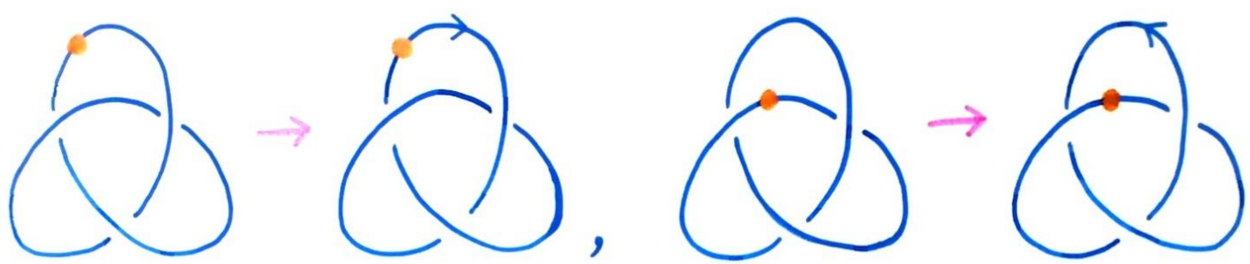
$X_{-D}(t)=4t^2$
$d(-D)=2$

- n:odd

From D, we can obtain an alternating diagram D' with even crossings as follows:



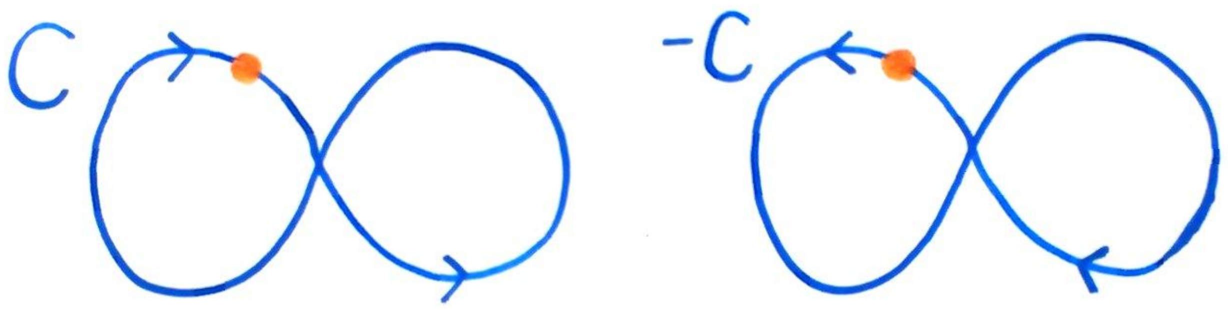
Example.



Then D', D, and C are oriented. □

Corollary 13.

For every based oriented curve C_b, there is no orientation-preserving, base-point preserving homeomorphism $\boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^2$ sending C_b to $-C_b$.



Thank you!