Seifert fibered surgeries with distinct primitive/Seifert positions

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joint with

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Seifert surgeries and primitive/Seifert positions

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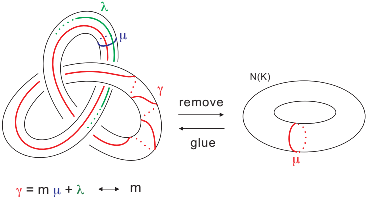

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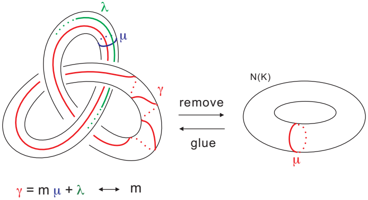

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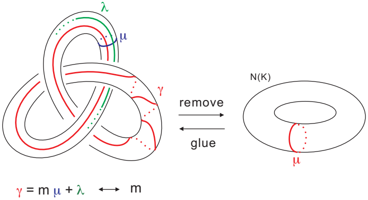

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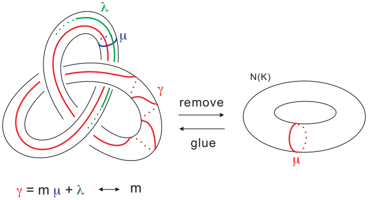

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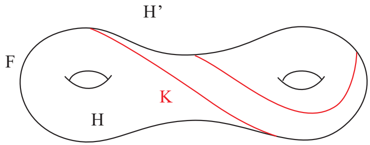
- *Does every Seifert surgery (K, m) have a primitive/Seifert position?.*
- *If (K, m) has such a position, then is it unique?*

Let K be a knot contained in a genus 2 Heegaard surface F which splits S^3 into two genus 2 handlebodies H and H', i.e. $S^3 = H \cup_F H'$.



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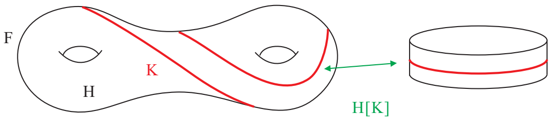

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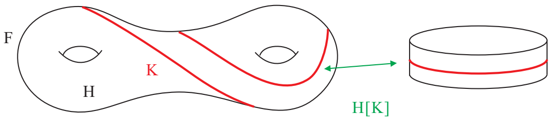

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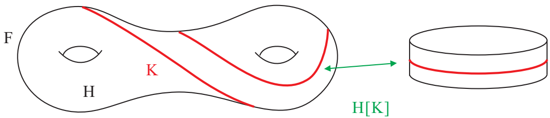

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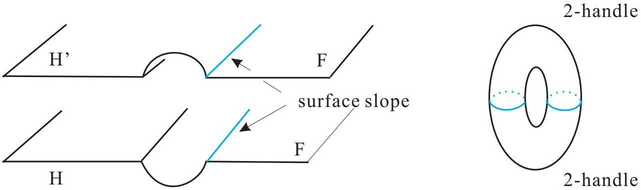
$H[K]$ is a solid torus $\leftrightarrow$ K is primitive w.r.t. H

and

$H'[K]$ is a solid torus $\leftrightarrow$ K is primitive w.r.t. H'.

Then by performing Dehn surgery on K along the surface slope m, we obtain a 3-manifold $K(m) = H[K] \cup H'[K]$, which is a lens space.



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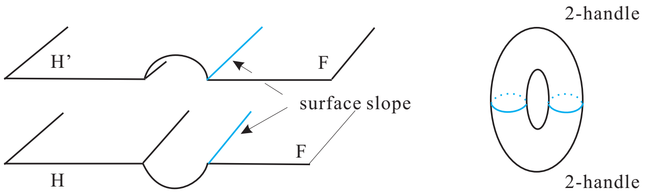

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We say that (K, m) has a primitive/primitive position (F, K, m)

Berge–Gordon conjecture

(K, m) is a lens surgery

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Berge announces that primitive/primitive position for a lens surgery (K, m) is "essentially" unique.

Suppose:

$H[K]$ is a solid torus $\leftrightarrow$ K is primitive w.r.t. H

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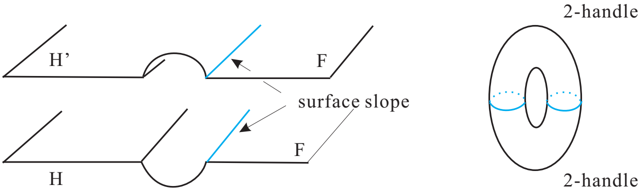
$H'[K]$ is a Seifert fiber space $\leftrightarrow$ K is Seifert w.r.t. H'.

Then by performing Dehn surgery on K along the surface slope m, we obtain a 3-manifold $K(m) = H[K] \cup H'[K]$, which is a Seifert fiber space or a connected sum of lens spaces.



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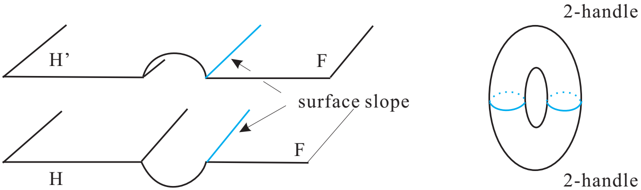

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Existence problem

There are infinitely many Seifert surgeries each of which does not have a primitive/Seifert position.
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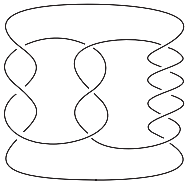
The simplest example is $(P(-3,3,5),1)$.



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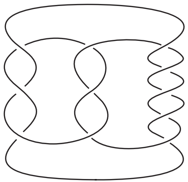

Idea: If a Seifert surgery (K, m) has a primitive/Seifert position, then K is strongly invertible.
On the contrary, knots in the above examples are not strongly invertible.

Uniquness problem

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Let (K, m) be a Seifert surgery which has primitive/Seifert positions (F_1, K_1, m) and (F_2, K_2, m); F_i is a genus 2 Heegaard surface and $K_i \subset F$ is isotopic to K in S^3.

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We say that two primitive/Seifert positions (F_1, K_1, m) and (F_2, K_2, m) are the same if there is an orientation preserving diffeomorphism f of S^3 satisfying $f(F_1) = F_2$ and $f(K_1) = K_2$.

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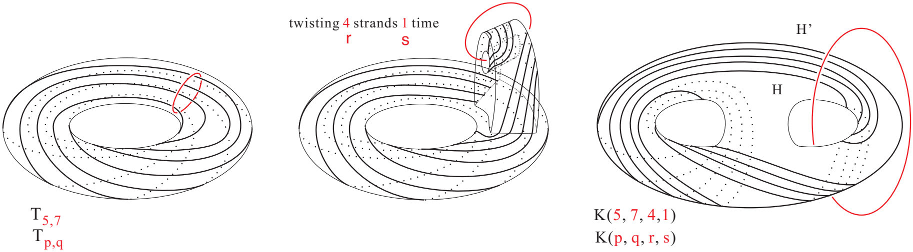
otherwise they are distinct.

Dean introduced twisted torus knots, which are naturally embedded in a genus 2 Heegaard surface.



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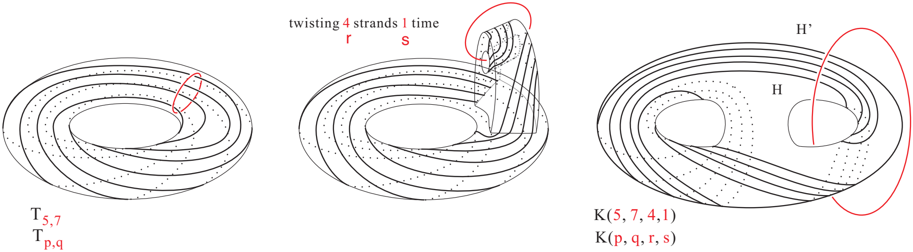

There are infinitely many twisted torus knots in primitive/Seifert positions.

Guntel's family

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Idea (along an example):
(1) Among Dean's twisted torus knots Guntel finds a pair of twisted torus knots $K(17, 5, 2, -1)$, $K(18, 5, 3, -1)$ in primitive/Seifert positions, which have the same surface slope 81 and also the same resulting Seifert fiber space $S^2(2, 3, 5)$.
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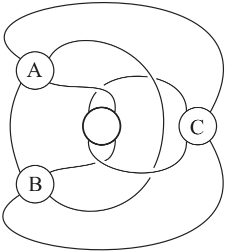
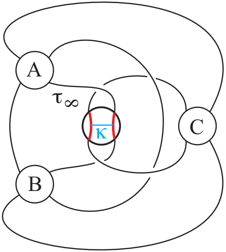
(2) Prove that $K(17, 5, 2, -1)$ and $K(18, 5, 3, -1)$ are actually isotopic in S^3 using conjugacy of elements in the braid group.



$\mathcal{B}$(A, B, C)

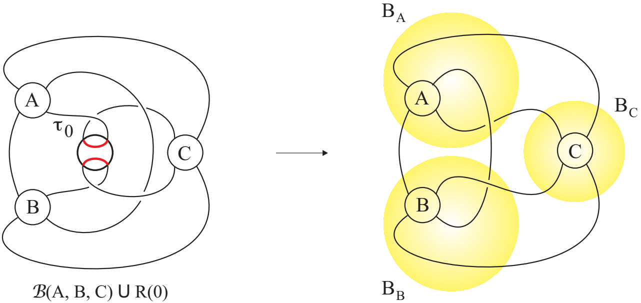


$\mathcal{B}$(A, B, C) ∪ R(∞)

Suppose that $A = R(l)$, $B = R(m, -l)$, $C = R(-n, 2, m-1, 2, 0)$.

Then $\mathcal{B}(A, B, C) \cup R(\infty)$ is a trivial knot in S^3.

$p : S^3 \to S^3$: two-fold branched covering branched along $\mathcal{B}(A, B, C) \cup R(\infty)$.
$K = K(l, m, n) = p^{-1}(\kappa)$ is a knot in S^3 (upstairs).



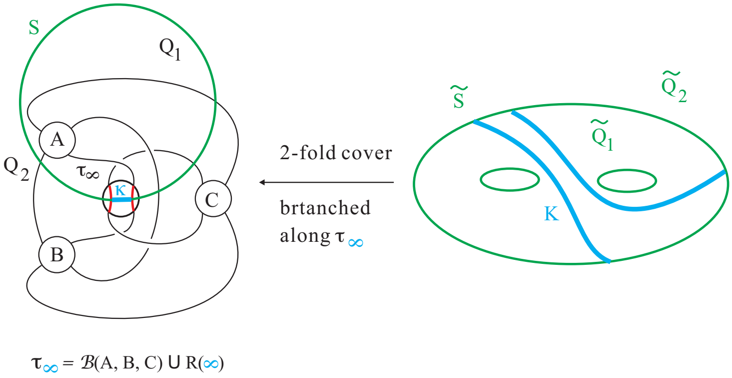
$\mathcal{B}(A, B, C) \cup R(0)$ is a Montesinos link with three rational tangles.

$$\begin{array}{ccc} S^3 & \xrightarrow{\gamma\text{–surgery along } K} & K(\gamma_0) \\ \text{ld branched cover} \downarrow & & \downarrow \text{2–fold branched cover} \\ \mathcal{B}(A,B,C) \cup R(\infty) & \xrightarrow[\text{0–untangle surgery along } \kappa]{} & \mathcal{B}(A,B,C) \cup R(0) \end{array}$$

DIAGRAM. Montesinos trick

Since $\mathcal{B}(A,B,C) \cup R(0)$ is a Montesinos link, $K(\gamma_0)$ is a Seifert fiber space.

Genus two Heegaard surfaces



S: 2–sphere bounding 3–balls Q_1 and Q_2.
$(Q_i, Q_i \cap \tau_\infty)$ is a 3–string trivial tangle.

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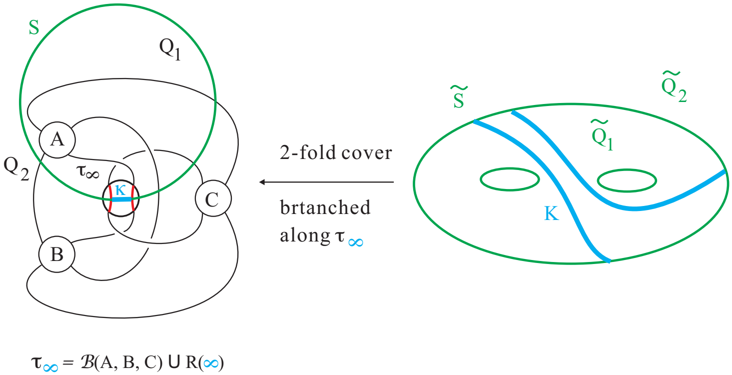

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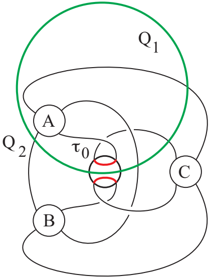
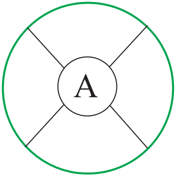
$$\Rightarrow \quad S^3 = \widetilde{Q_1} \cup_{\widetilde{S}} \widetilde{Q_2}, \quad K \subset \widetilde{S}.$$

$\widetilde{S}$ is a genus two Heegaard surface carrying K.

Genus two Heegaard surfaces



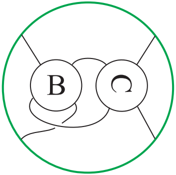
$\tau_0 = \mathcal{B}(A, B, C) \cup R(0)$



rational tangle

2-cover ← branched along τ_0

$\widetilde{Q}_1 \cup$ 2-handle
$\|$
solid torus



Montesinos tangle

2-cover ← branched along τ_0

$\widetilde{Q}_2 \cup$ 2-handle
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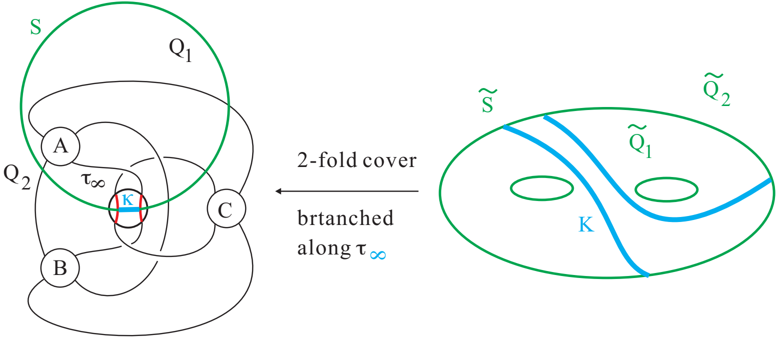
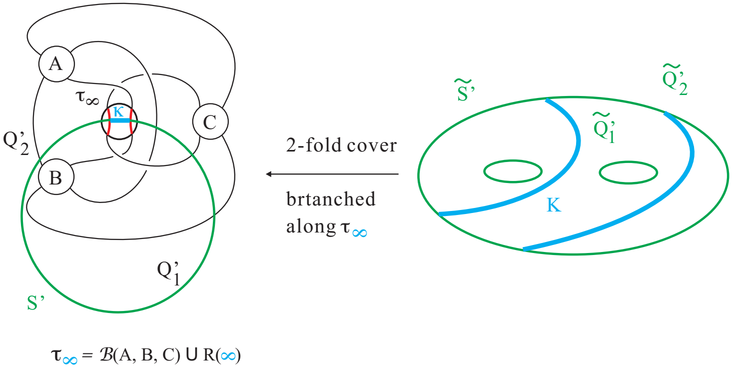
Seifert fiber space
$D^2(p, q)$

Primitive/Seifert postion



$\tau_\infty = \mathcal{B}(A, B, C) \cup R(\infty)$

$(\widetilde{S}, K, m)$ is a primitive/Seifert position of (K, m).



S': 2–sphere bounding 3–balls Q'_1 and Q'_2.
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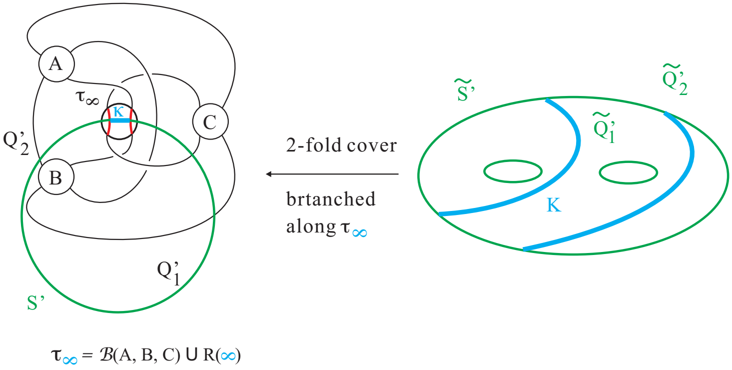

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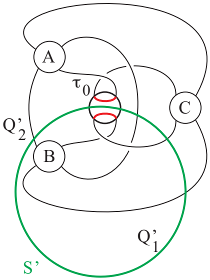
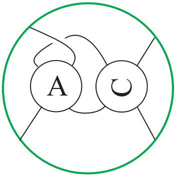
$\Rightarrow \ S^3 = \widetilde{Q'_1} \cup_{\widetilde{S'}} \widetilde{Q'_2}, \quad K \subset \widetilde{S}.$

$\widetilde{S'}$ is a genus two Heegaard surface carrying K.

Genus two Heegaard surfaces



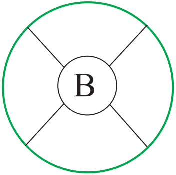
$\tau_0 = \mathcal{B}(A, B, C) \cup R(0)$



Montesinos tangle

2-cover ← branched along τ_0

$\widetilde{Q}'_2 \cup$ 2-handle
||
Seifert fiber space $D^2(p, q)$



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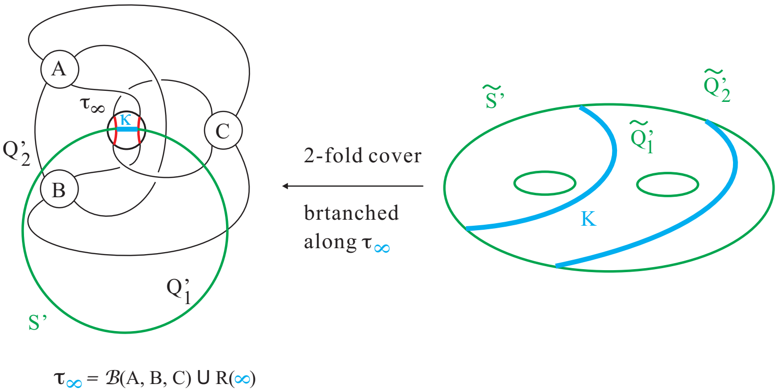
solid torus



$(\widetilde{S}', K, m)$ is a primitive/Seifert position of (K, m).

Compare $(\widetilde{S}, K, m)$ with $(\widetilde{S}', K, m)$

Now we have two primitive/Seifert positions:

$(\widetilde{S}, K, m)$ and $(\widetilde{S}', K, m)$.

Compare $(\widetilde{S}, K, m)$ with $(\widetilde{S'}, K, m)$

Now we have two primitive/Seifert positions:

$(\widetilde{S}, K, m)$ and $(\widetilde{S'}, K, m)$.

We need to show that they are distinct .

Invariant for Primitive/Seifert postions

Let (F, K, m) be a primitive/Seifert position for (K, m).

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Assume:

K is primitive w.r.t. V, i.e. $V[K]$ is a solid torus

K is Seifert w.r.t. W, i.e. $W[K]$ is a Seifert fiber space $D^2(p, q)$.

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Define: $i(F, K, m) = \{p, q\}$

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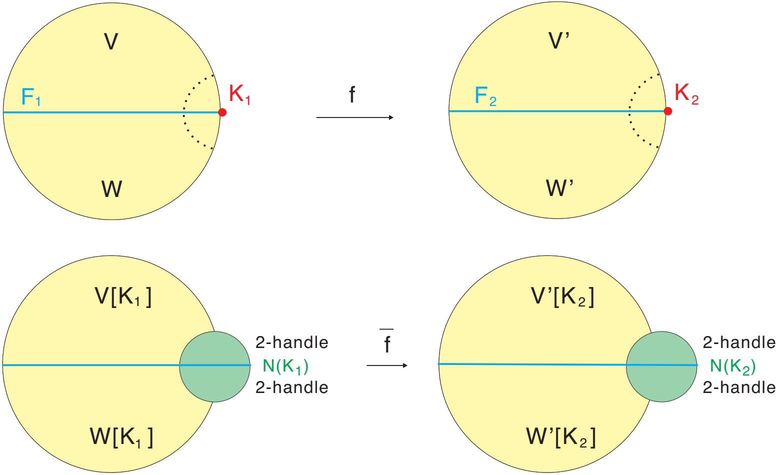
Lemma

Two primitive/Seifert positions (F_1, K_1, m) and (F_2, K_2, m) for a Seifert fibered surgery (K, m) are the same
$\Rightarrow$
$i(F_1, K_1, m) = i(F_2, K_2, m)$.

Proof:


V
F1
K1
W
f
V'
F2
K2
W'
V[K1]
2-handle
N(K1)
2-handle
W[K1]
f̄
V'[K2]
2-handle
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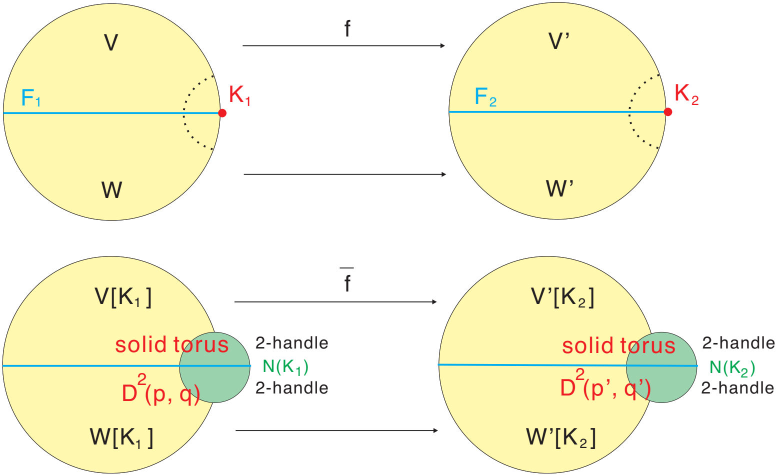
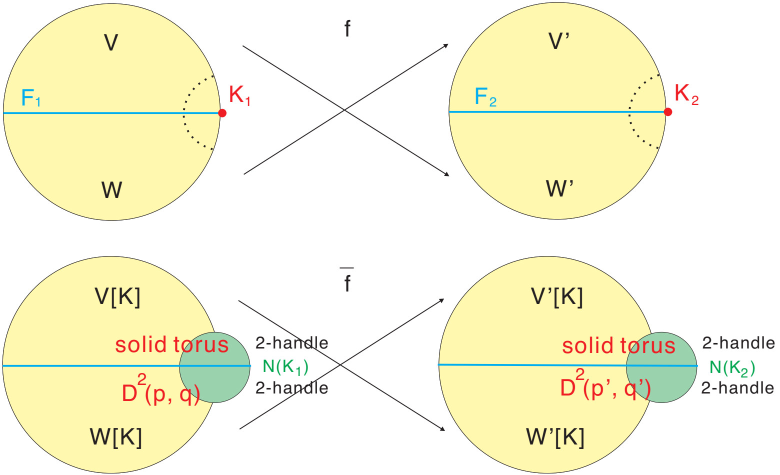
2-handle
W'[K2]

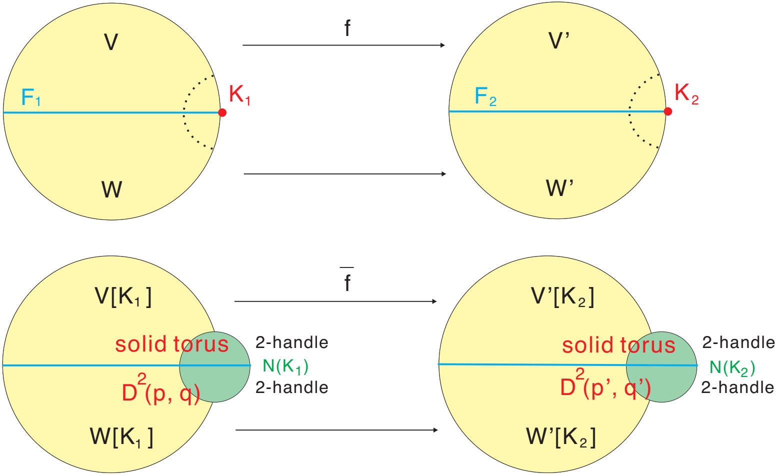
Invariant for Primitive/Seifert postions



Invariant for Primitive/Seifert postions

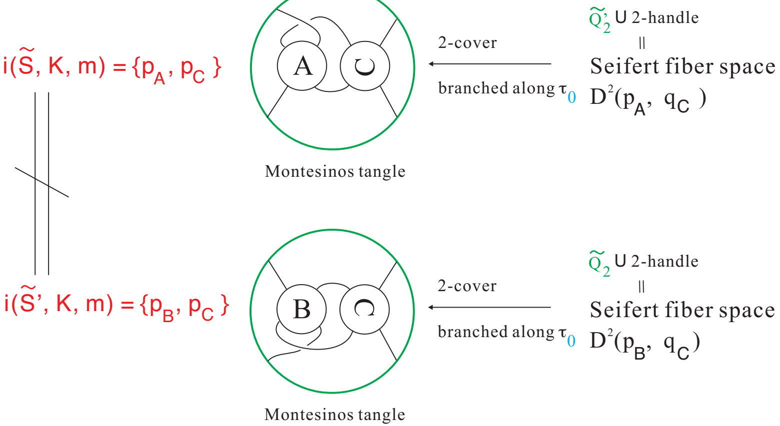


Invariant for Primitive/Seifert postions



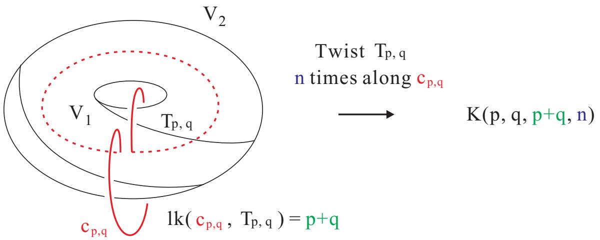
$D^2(p, q) \cong D^2(p', q') \ \Rightarrow \{p,\ q\} = \{p',\ q'\}$
$\Rightarrow i(F_1, K_1, m) = i(F_2, K_2, m)$

Invariant for Primitive/Seifert postions



Hence $(\widetilde{S}, K, m)$ and $(\widetilde{S}', K, m)$ are distinct primitive/Seifert positions for (K, m).

Choose twisted torus knots $K(p, q, p+q, n)$ $(|p+q| \neq 1)$.



Theorem

1. $K(p, q, p+q, n)(pq + n(p+q)^2)$ *is a Seifert fiber space* $S^2(|p|, \ |q|, \ |n|)$.
2. *For each relatively prime integers* p, q *and* n $(|n| \geq 2)$, $K(p, q, p+q, n)$ *has distinct primitive/Seifert positions.*

Hyperbolic knots with arbitrarily large volume

Since we can choose $K(p, q, p + q, n)$ so that $S^3 - K(p, q, p + q, n)$ has an arbitrarily large volume, we have:

Corollary

For any $r > 0$, there is a Seifert surgery (K, m) on a hyperbolic knot K such that

1. (K, m) *has two distinct primitive/Seifert positions, and*
2. $\mathrm{Vol}(K) > r$.

Questions

Suppose: (K, m) has more than one primitive/Seifert position

- Can we explain such a phenomenon geometrically?

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At most three?

$K(m) = S^2(p, q, r)$