Cosmetic surgeries and non-orientable surfaces

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1. Introduction

Knot Complement Conjecture

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conjectured by Tietze (1908)
proved by Gordon-Luecke (1989)

Gordon-Luecke's result

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Theorem (Gordon-Luecke, 1989)

On a nontrivial knot in S^3,
nontrivial Dehn surgery never yields S^3.

Remark:

A similar result for $S^2 \times S^1$ is obtained by Gabai (1987).

Dehn surgery

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(i.e., $M-$(open tubular nbd of K))

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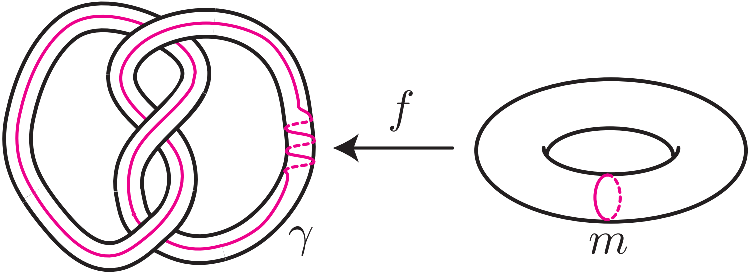
Gluing solid torus V to $E(K)$ along slope γ



slope $=$ isotopy class of
non-trivial unoriented simple loop

Parametrization of slope

For a knot K in the 3-sphere S^3,
by using a meridian-longitude system,
we have a correspondence:

$$\{\text{slope on } \partial N(K)\} \overset{1:1}{\longleftrightarrow} \left\{ \frac{p}{q} \right\} \cup \left\{ \frac{1}{0} \right\}$$

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Example

$1/0$ represents the meridian
(i.e., bounding a disk in the nhd. of the knot),
0 represents the preferred longitude
(i.e., bounding a surface in the knot exterior).

Generalization

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Oriented Knot Complement Conjecture (Bleiler (Kirby's list Problem 1.81(D)))

If K_1 and K_2 are knots in a closed, oriented 3-manifold M whose complements are homeomorphic via an orientation-preserving homeomorphism, then there exists an orientation-preserving homeomorphism of M taking K_1 to K_2.

Cosmetic surgery

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Cosmetic surgery conjecture (Bleiler (Kirby's list Problem 1.81(A)))

Two surgeries on inequivalent slopes are never purely cosmetic.

i.e., if $K(r_1) \cong K(r_2)$ for inequivalent slopes, then the homeo. is orientation reversing.

Definition

We call

- two slopes are equivalent
  if there exists a homeomorphism of $E(K)$
  taking one slope to the other.

- two surgeries on K along slopes r_1 and r_2
  are purely cosmetic
  if there is an orientation preserving
  homeomorphism between $K(r_1)$ and $K(r_2)$,
  and chirally cosmetic
  if the homeo. is orientation reversing.

Remarks

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Actually
$(18k+9)/(3k+1)$- and
$(18k+9)/(3k+2)$-surgeries
on the trefoil knot $T_{2,3}$ in S^3 yield
orientation-reversingly homeomorphic pairs
for any non-negative integer k.

2. Known Facts

Known facts (Examples)

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Seifert knots in closed 3-mfds

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Non-hyperbolic knots in lens spaces admitting cosmetic surgeries are classified.

They are all chirally cosmetic.

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Theorem [Bleiler-Hodgson-Weeks (1999)]

There exists a hyperbolic knot which admits a pair of surgeries along inequivalent slopes yielding oppositely oriented lens spaces.

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$$L(49, -19) \leftrightarrow L(49, -18) \quad \text{(mirror image)}$$

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$L(49, -19) \leftrightarrow L(49, -18)$ (mirror image)

It was announced by Matignion (preprint) that there are infinite family extended above.

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Theorem [Boileau-Domergue-Mathieu (1995)]

Non-trivial surgery on a null-homotopic knot K never yields the original manifold.

Known facts (Bounds)

Theorem [Lackenby (1997)]

Let K be a homotopically trivial knot with irreducible, atoroidal exterior in 3-mfd with $\beta_1 > 0$.
Suppose that at least one of the slopes r, r' has a sufficiently high distance with the meridian.
Then $K(r)$ and $K(r')$ are orientation preserving homeomorphic if and only if $r = r'$, and are orientation reversing homeomorphic if and only if K is amphicheiral and $r = -r'$.

Known facts (Bounds)

Theorem [Bleiler-Hodgson-Weeks (1999)]

For a hyperbolic knot K,
there exists a finite set of slopes E
such that if r, r' are distinct slopes outside E,
$K(r)$ and $K(r')$ homeomorphic implies that
${}^{\exists}h$ orientation reversing isometry s.t. $h(r) = r'$.
In particular, if K is a knot in S^3,
then K is amphicheiral and $r = -r'$.

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Theorem [Wu (2011)]

Let r and r' be two distinct rational numbers with $rr' > 0$, and K a non-trivial knot in S^3. Then $K(r) \not\cong K(r')$ (orientation preservingly).

Recent Progress

Theorem [Ni-Wu (2011)]

Suppose K is a nontrivial knot in S^3, $r_1, r_2 \in \mathbb{Q} \cup \{0/1\}$ are two distinct slopes such that $K(r_1) \cong K(r_2)$ as oriented manifolds. Then r_1, r_2 satisfy that

(a) $r_1 = -r_2$;

(b) suppose $r_1 = p/q$, then $q^2 \equiv -1 \pmod{p}$;

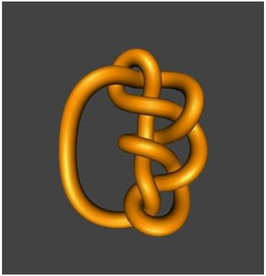
(c) $\tau(K) = 0$, where τ is the invariant defined by Ozsváth-Szabó.

3. Result

Theorem

Let $K = 9_{27}$ in knot table.
Then $K(10/3) \not\cong K(-10/3)$.

In fact, $9_{27} = S(49, 19)$ (2-bridge knot).



Is it new?

Let $K = 9_{27}$.

- $\Delta_K(t) = -t^3 + 5t^2 - 11t + 15 - 11t^{-1} + 5t^{-2} - t^{-3}$
  $\Rightarrow \quad \Delta''_K(1) = 0.$
- $p/q = 10/3$
  $\Rightarrow q^2 = 9 \equiv -1 (\mod p = 10).$
- $K = 9_{27}$ is a slice knot
  $\Rightarrow |\tau(K)| \leq g_4(K) = 0.$

  Actually K is a 2-bridge knot
  $\Rightarrow \tau(K) = \sigma(K)$ (signature)

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Fact

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If p is even, $K(p/q)$ contains a closed non-orientable surface.

Let us consider the minimal genus of such non-ori surfaces embedded in $K(10/3)$ and $K(-10/3)$ for $K = 9_{27}$.

Convention: for non-ori surf F

genus of F := $\sharp$ of Mobius bands in F

$K(-10/3)$

Claim 1

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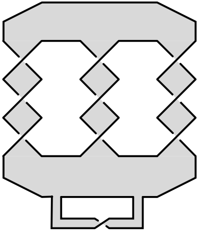
$\Rightarrow$ $^{\exists}F_1$: non-ori. surf of genus 5 with ∂-slope $-10/3$



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Claim 2

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Proposition (Przytycki, 1983)

$\hat{F}_2$ can be isotoped so that
$\hat{F}_2 \cap E(K) =: F_2$ is
incomp, ∂-incomp, not ∂-parallel in $E(K)$.

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There are exactly 8 such surfaces in $E(K)$

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However $\hat{F}_2 - F_2 =$(disks)
implies ∂-slope must be $10/3$, contradiction.

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F_2': non-ori incomp, ∂-comp. surf in $E(K)$
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$\Rightarrow$ **$\Delta(r_2, 10/3)$ must be 2**. contradiction.

$\square$