複素近傍の分類理論と Levi 葉層

Classification of complex neighborhoods and Levi-flat foliations

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Classifications of complex neighborhoods

X: complex surface (possibly noncompact)

Y: compact Riemann surface in X, non-singular

V: (a sufficiently small) open nbhd of Y in X

Problem

What kinds of complex analytic structures appear in V?

- existence of holomorphic/(s.)p.s.h./p.h. functions on V or $V \setminus Y$

What kinds of geometric structures appear in V??

- existence of holomorphic foliations with compact leaf Y
- "degeneration" of the family of Levi-flat hypersurfaces or contact type hypersufaces

Rmk.

"The tubular neighborhood theorem" does **NOT** *hold in holomorphic setting.*

Known Results ([Grauert'62], [Suzuki'75], [Ueda'83])

Theorem (Grauert '62)

If $c_1(N_{Y/X}) < 0$, then there exist a neighborhood V of Y and

a s.p.s.h.function $\Phi : V \setminus Y \to \mathbb{R}$ s.t. $\Phi(p) \to -\infty$ $(p \to Y)$,

that is, there is a strongly pseudoconvex neighborhood system around Y.

In particular, ∂V is a contact manifold.

- Φ: strictly plurisubharmonic (s.p.s.h) function if
  $-dJ^*d\Phi(v, Jv) = \sqrt{-1}\partial\bar{\partial}\Phi(v, Jv) > 0$ for $\forall v \in TV, v \neq 0$
  in other words, $\Delta\Phi|_D > 0$ for any holomorphic disk D in V.
- $\alpha := -J^*d\Phi|_{\partial V}$ defines a positive contact structure ξ on ∂V.
  $\xi = \mathrm{Ker}\ \alpha$ is J-invariant, $\xi = T(\partial V) \cap JT(\partial V)$

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- $\alpha := -J^*d\Phi|_{\partial V}$ defines a (Levi-flat) foliation ξ on ∂V.

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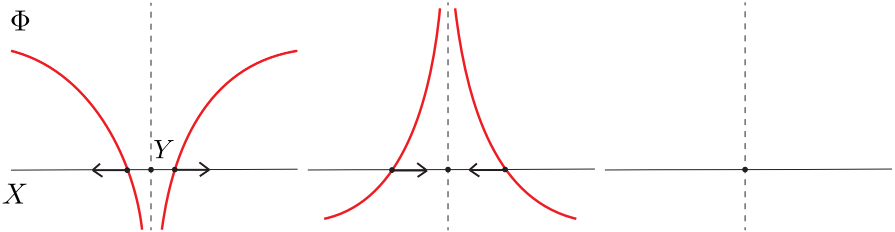
Known Results ([Grauert'62], [Suzuki'75], [Ueda'83])

- $c_1(N_{Y/X}) < 0 \iff \exists$ strongly pseudoconvex nbhd system around Y
- $c_1(N_{Y/X}) > 0 \implies \exists$ strongly pseudoconcave nbhd system around Y
- $c_1(N_{Y/X}) = 0 \quad \cdots$

$c_1(N_{Y/X}) < 0$ $\qquad$ $c_1(N_{Y/X}) > 0$ $\qquad$ $c_1(N_{Y/X}) = 0$



Q. How about the case $c_1(N_{Y/X}) = 0$?

Ueda's neighborhood theory

Observations in the case $c_1(N_{Y/X}) = 0$

Ex.1 (Product case) ($\exists$ pseudo-flat nbhd system)

Y: compact Riemann surface,

$X = Y \times \mathbb{C}P^1 \supset Y = Y \times \{[1:0]\}$, $N_{Y/X} = \mathbf{1}_Y$ (holomorphically)

$\Phi : V \backslash Y \to \mathbb{R}$, $\Phi(z, w) = \pm \log |w|$ p.h.function, $\Phi(p) \to \mp\infty$ $(p \to Y)$

$\Phi^{-1}(c) =$ Levi-flat hypersurface

Ex.2 (Serre's example) ($\exists$ pseudo-convave nbhd system)

$$X = \mathbb{C}^2_{(z,w)} / \sim \ \cup \ Y$$

where $(z, w) \sim (z+1, w+1) \sim (z+\tau, w+\overline{\tau})$ and Y elliptic curve

$X = Y \times_\rho \mathbb{C}P^1 \supset Y = Y \times \{[0:1]\}$, $N_{Y/X} = \mathbf{1}_Y$ (holomorphically)

$\Phi : V \backslash Y \to \mathbb{R}$, $\Phi(z, w) = |w - \bar{z}|^2$ s.p.s.h.function, $\Phi(p) \to \infty$ $(p \to Y)$

$\Phi^{-1}(c) =$ contact type hypersurface, $\exists \mathcal{F}$: holo.foliation with leaf Y

Observations in the case $c_1(N_{Y/X}) = 0$

Ex.3 (Ueda's example [Ueda'83], Koike's example [Koike'15])

- $\exists$ (Y, X) with $c_1(N_{Y/X}) = 0$, $\forall V$ nbd of Y,

  $\nexists$ $\Phi : V \backslash Y \to \mathbb{R}$ p.s.h.function s.t. $\Phi(p) \to \infty$ $(p \to Y)$.

In [U'83], Ueda classified into the following three cases $\cdots$ type α, β, γ.

- type α $\cdots$ "$\exists$ pseudo-concave neighborhood system"
- type β $\cdots$ $\exists$ pseudo-flat neighborhood system
- type γ $\cdots$ otherwise

Ex.1$\cdots$ type β, Ex.2$\cdots$ type α, Ex.3$\cdots$ type γ

Main result (foliations v.s. Ueda type)

Y: elliptic curve, X: tubular neighborhood of Y

$\mathcal{F}$: holomorphic foliation in X with the compact leaf Y

holonomy of $\mathcal{F}$ along Y: $\rho : \pi_1(Y, *) \to \mathrm{Diff}(\mathbb{C}, 0)$

$$f(z) = \lambda z + O(z^2), \quad g(z) = \mu z + O(z^2)$$

We assume that $\lambda, \mu \in \mathrm{U}(1)$.

		λ: torsion		λ: non-torsion	
		f:lin'ble	f:non-lin'ble	f:lin'ble	f:non-lin'ble
μ tor	g:lin	I	II	III	IV
	g:non-lin		V	VI	VII
μ non-tor	g:lin			VIII	IX
	g:non-lin				X

Q. Which type appears in I $\sim$X ?

Ueda's classification – type (β)

Setup

- X: complex manifold of $\dim_{\mathbb{C}} = 2$,
- $Y \subset X$: compact (non-singular) Riemann surface holomorphically embedded in X with $c_1(N_{Y/X}) = 0$.

Definition (type (β))

The pair (Y, X) is said to be *of type* (β) if there exists a (non-singular) holomorphic foliation $\mathcal{F}$ defined on a neighborhood of Y which also has Y as a leaf and has $U(1)$-*linear holonomy* along Y
(i.e. the image of the holonomy function $\mathrm{Hol}_{\mathcal{F},Y} : \pi_1(Y, *) \to \mathrm{Diff}(\mathbb{C}, 0)$ is a subgroup of $U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$).

Observation: (Y, X): of type (β) if Y admits a holomorphic tubular neighborhood ($\Leftarrow$ the fact that $N_{Y/X}$ admits $U(1)$-flat connection).

Idea of Ueda's classification theory

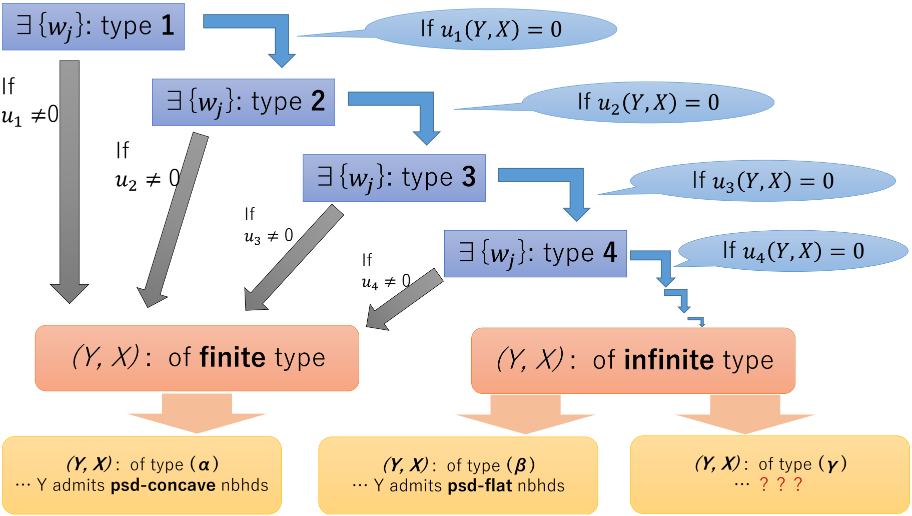
Idea

Classify (Y, X) in accordance with the difference from "the case of type (β)" in n-jet sense (along Y, $n \in \mathbb{Z}_{>0}$).

In what follows, we will try to explain Ueda's classification theory in the following steps:

Step 1: Alternative definition of type (β) by using local defining functions
Step 2: The notion "local defining functions of type n"
Step 3: Ueda's obstruction class $u_n(Y, X)$
Step 4: Definition of type (α) and (γ)
Step 5: Ueda's theorems on the classification

Summary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If $u_1(Y,X) = 0$
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(Y, X) : of **finite** type
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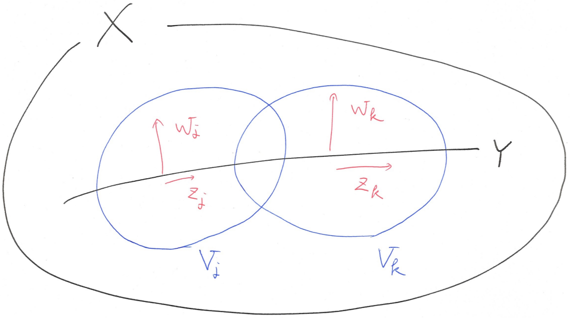
(Y, X): of type $(\boldsymbol{\alpha})$
… Y admits **psd-concave** nbhds
(Y, X): of type $(\boldsymbol{\beta})$
… Y admits **psd-flat** nbhds
(Y, X): of type $(\boldsymbol{\gamma})$
… ? ? ?

Step1: Alternative definition of type (β)

Take an open covering $\{V_j\}$ of a small neighborhood V of Y and a holomorphic coordinates system (z_j, w_j) of each V_j as follows:



- z_j: an extension of a coordinate z_j on $V_j \cap Y$
- w_j: a local defining function of $V_j \cap Y$

Step1: Alternative definition of type (β) (continuation)

From the following, we may assume that there exists $t_{jk} \in U(1)$ such that

$$\left.\frac{w_j}{w_k}\right|_{V_{jk}\cap Y} \equiv t_{jk}$$

holds on each $V_{jk} := V_j \cap V_k$.

Theorem

Let Y be a compact Kähler manifold and N be a line bundle on Y. Assume that $c_1(N) = 0$. Then N is $U(1)$-flat (i.e. the transition functions $\in U(1)$ for a suitable choice of a local trivialization of N).

Therefore, we have the following form of the expansion of the function $t_{jk}w_k|_{V_{jk}}$ by w_j:

$$t_{jk}w_k = w_j + f_{jk}^{(2)}(z_j)\cdot w_j^2 + f_{jk}^{(3)}(z_j)\cdot w_j^3 + f_{jk}^{(4)}(z_j)\cdot w_j^4 + O(w_j^5)$$

Step1: Alternative definition of type (β) (continuation)

Definition (alternative definition of type (β))

The pair (Y, X) is said to be *of type* (β) if

$$t_{jk} w_k = w_j$$

holds on each V_{jk} by choosing w_j's appropriately.

c.f. $\mathcal{F} :=$ "$\{w_j \equiv (\text{constant})\}$"

Definition (type (β), repeated)

The pair (Y, X) is said to be *of type* (β) if there exists a (non-singular) holomorphic foliation $\mathcal{F}$ defined on a neighborhood of Y which also has Y as a leaf and has $U(1)$-*linear holonomy* along Y.

Step2: Local defining functions of type n

$\{(V_j, (z_j, w_j))\}$: as above ($t_{jk}w_k = w_j + f^{(2)}_{jk}(z_j) \cdot w_j^2 + \cdots$).

Definition (Local defining functions of type n)

$\{w_j\}$ is said to be *of type* n if, for any $\nu \leq n$, it holds that $f^{(\nu)}_{jk} \equiv 0$ for each j, k.

i.e.

$$t_{jk}w_k = w_j + f^{(n+1)}_{jk}(z_j) \cdot w_j^{n+1} + f^{(n+2)}_{jk}(z_j) \cdot w_j^{n+2} + \cdots$$

holds for $\{w_j\}$ of type n.

$\exists\{w_j\}$ of type $n \Leftrightarrow$ "(Y, X) seems to be type (β) in n-jet along Y"

Note: Our $\{w_j\}$ is always at least of type 1.

Step3: Ueda's obstruction class $u_n(Y, X)$

Assume that $\exists\{w_j\}$ of type n. One can deduce from

$$t_{jk}w_k = w_j + f_{jk}^{(n+1)}(z_j)\cdot w_j^{n+1} + f_{jk}^{(n+2)}(z_j)\cdot w_j^{n+2} + \cdots .$$

that

$$\left(\frac{1}{t_{jk}w_k}\right)^n = \left(\frac{1}{w_j}\right)^n \cdot (1 - f_{jk}^{(n+1)}(z_j)\cdot w_j^n + O(w_j^{n+1}))^n.$$

Therefore,

$$f_{jk}^{(n+1)}|_{V_j\cap Y} = \frac{1}{n}\left[\frac{1}{w_j^n} - t_{jk}^{-n}\frac{1}{w_k^n}\right]\Bigg|_{V_j\cap Y}.$$

From the calculation above, we have that

Prop.

$\left\{\left(V_j\cap Y, f_{jk}^{(n+1)}\right)\right\}$ *satisfies the* 1*-cocycle condition as sections of* $N_{Y/X}^{-n}$.

Step3: Ueda's obstruction class $u_n(Y, X)$ (continuation)

Definition

$u_n(Y, X) := \left[\left\{\left(V_j \cap Y, f_{jk}^{(n+1)}\right)\right\}\right] \in H^1(Y, N_{Y/X}^{-n})$: n-*th Ueda class.*

Here we denote by $H^1(Y, N_{Y/X}^{-n})$ the 1-st Čech cohomology group $\check{H}^1(Y, \mathcal{O}_Y(N_{Y/X}^{-n}))$ of the sheaf of holomorphic sections of $N_{Y/X}^{-n}$.

Key Prop.

(1) *When* $\exists\{w_j\}$ *of type* n*, the condition "*$u_n(Y, X) = 0$*" does not depend on the choice of* $\{w_j\}$ *of type* n*.*
(2) *Assume that* $\exists\{w_j\}$ *of type* n*. Then* $u_n(Y, X) = 0$ *iff* $\exists\{w_j\}$ *of type* $n + 1$*.*

Step4: Definition of type (α) and (γ)

By the Key Proposition in the previous page, only one of the following holds:

- $\exists n \in \mathbb{Z}_{>0}$ s.t. $\exists \{w_j\}$ of type n and $u_n(Y, X) \neq 0$.
- $\forall n \in \mathbb{Z}_{>0}$, $\exists \{w_j\}$ of type n and $u_n(Y, X) = 0$.

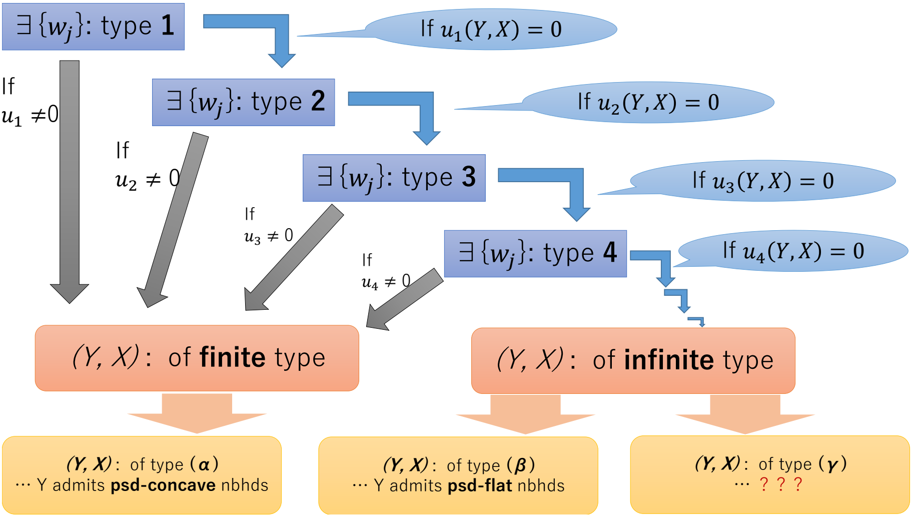
In the former case, (Y, X) is said to be *of finite type* (or more precisely, *of type* n).
In the latter case, (Y, X) is said to be *of infinite type*.

Note: (Y, X) of type $(\beta) \Rightarrow$ of infinite type.

Definition

(Y, X) is said to be *of type* (α) if it is of finite type.
(Y, X) is said to be *of type* (γ) if it is of infinite type however it is not of type (β).

Summary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$\cdots$? ? ?

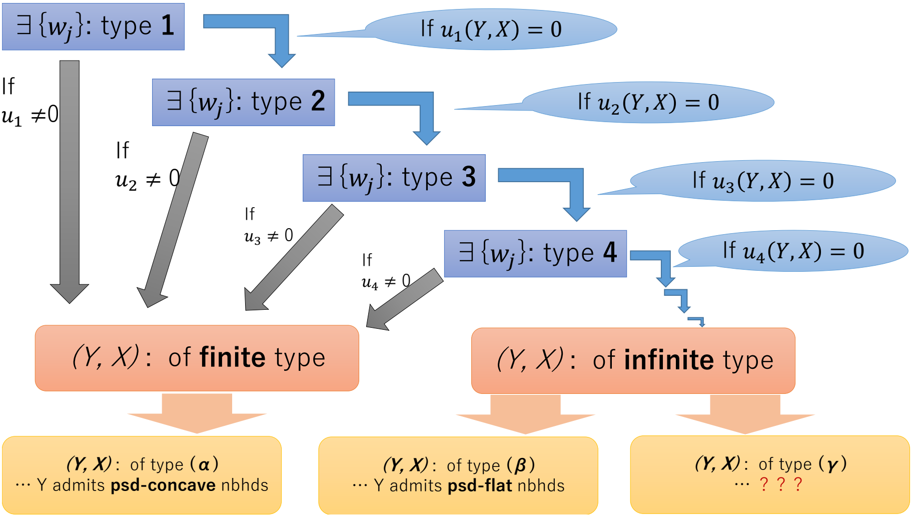
Step5: Ueda's theorems on the classification

Theorem (Ueda '83)

(1) $N_{Y/X} \in \mathrm{Pic}^0(Y)$: *torsion* $\Rightarrow$ (Y, X): *of type* (α) *or* (β).
(2) $N_{Y/X} \in \mathrm{Pic}^0(Y)$: *Diophantine (see below) and* (Y, X): *of infinite type* $\Rightarrow$ (Y, X): *of type* (β).
(3) (Y, X): *of type* (α) $\Rightarrow$ *there exists a* $\mathbb{R}$*-valued function* Φ *on a neighborhood* V *of* Y *s.t.* $\Phi|_{V\setminus Y}$*: s.p.s.h,* $\Phi(p) \to +\infty$ *as* $p \to Y$. *Especially,* Y *has a str. pseudoconcave neighborhoods system in this case.*
(4) $\exists$ *an example* (Y, X) *of type* (γ).

Observation: When (Y, X): of type (β), then there exists a $\mathbb{R}$-valued function Φ on a neighborhood V of Y s.t. $\Phi|_{V\setminus Y}$: pluriharmonic, $\Phi(p) \to +\infty$ as $p \to Y$ ($\Leftarrow \Phi(z_j, w_j) := \log|w_j|$). Especially, Y has a psudoflat neighborhoods system in this case.

Summary



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