高次元の接触構造のねじれと過旋性について

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Jan. 19, 2017, at 金沢大学サテライト・プラザ.
研究集会「 接触構造、特異点、微分方程式及びその周辺」

Goals

- To introduce the followings:
  - overtwistedness,
  - Plastikstufe,
  - Loose Legendrian submanifold.
- Result: ξ: contact structure on M^{2n+1}, $n > 1$.

<u>Theorem.</u> (A.) ξ: overtwisted $\Longleftrightarrow$

$\exists \mathcal{P} \subset (M, \xi)$:plastikstufe,

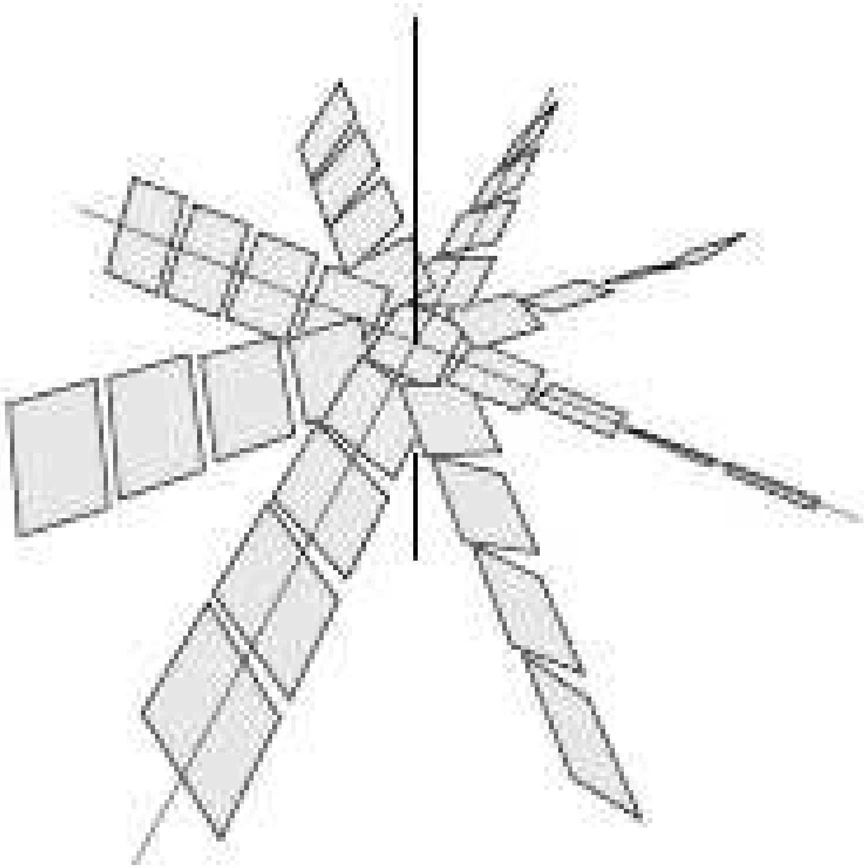
s.t. · small, · toric core, · trivial rotation.

- The basement of the ideas is the contact round surgery

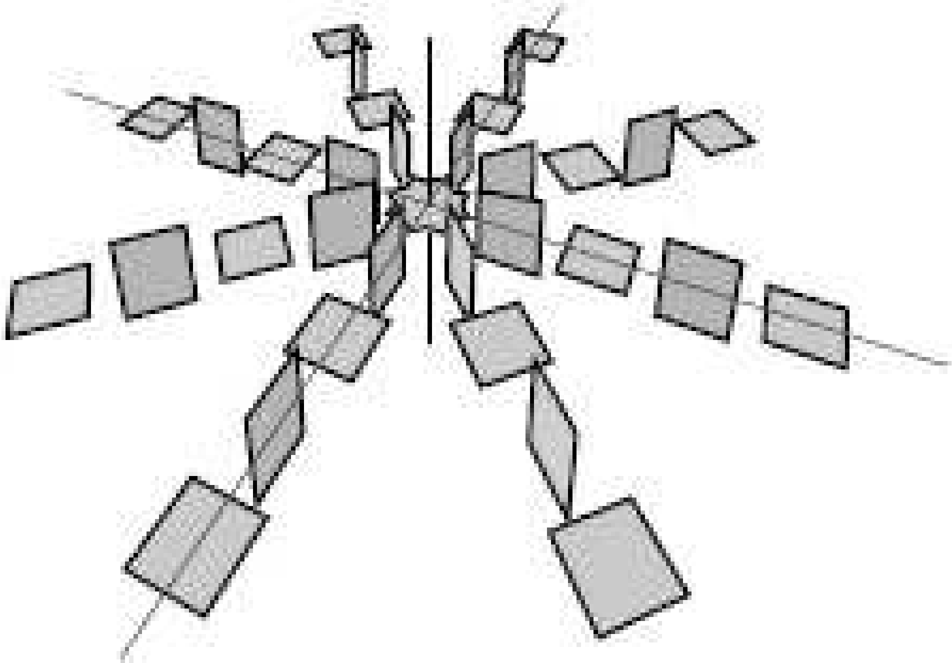
(A, 2014).

§ Overtwisted disc.

◎ In dimension 3



$$\ker\{dz + r^2 d\theta\}$$



$$\ker\{\cos r^2 dz + \sin r^2 d\theta\}$$

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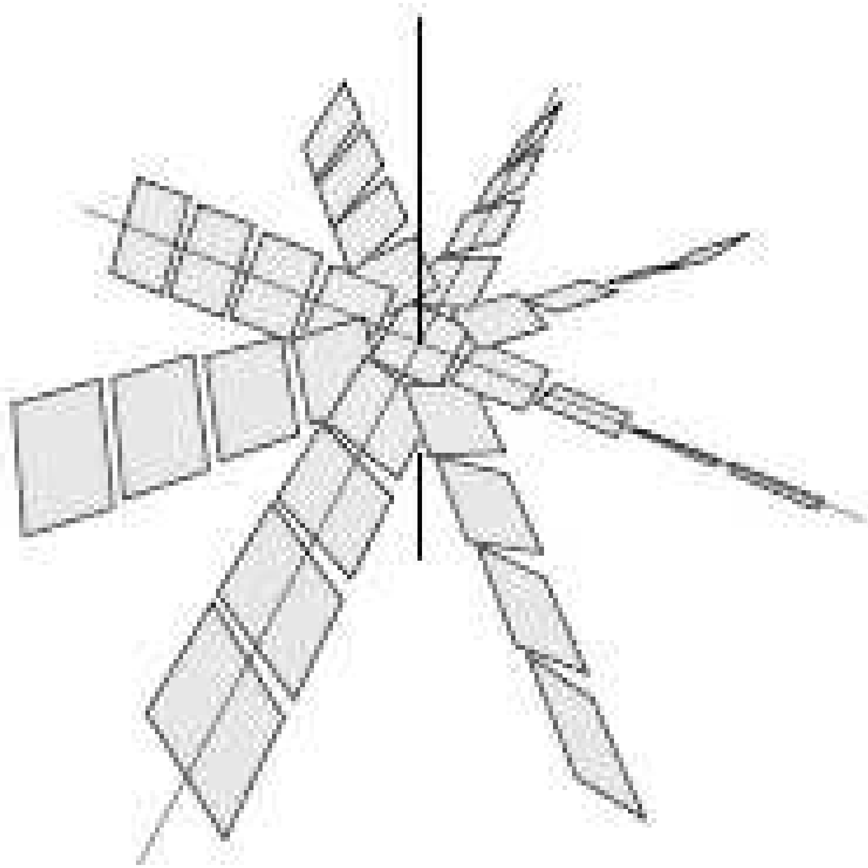

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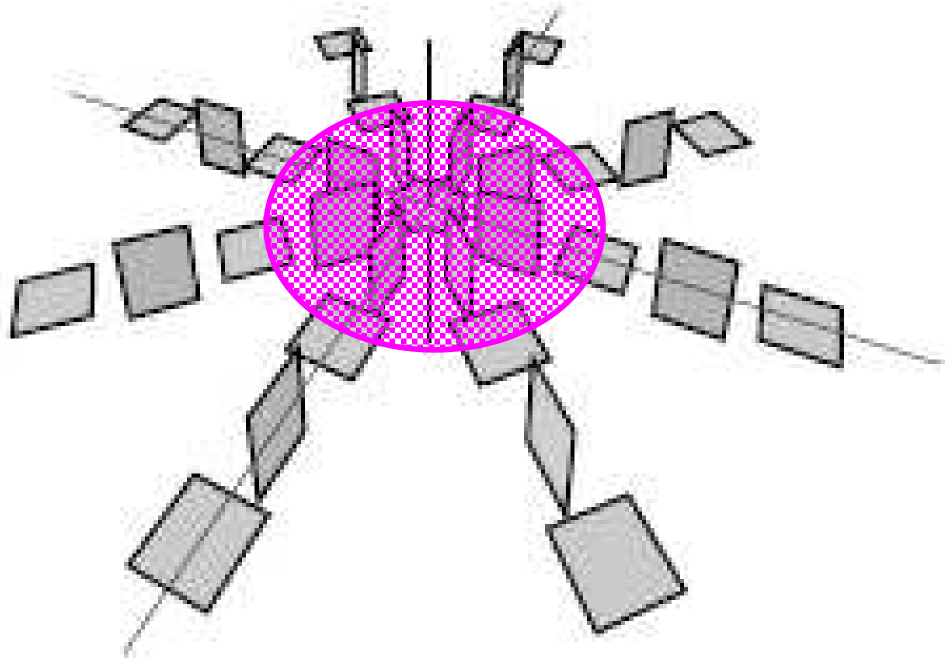

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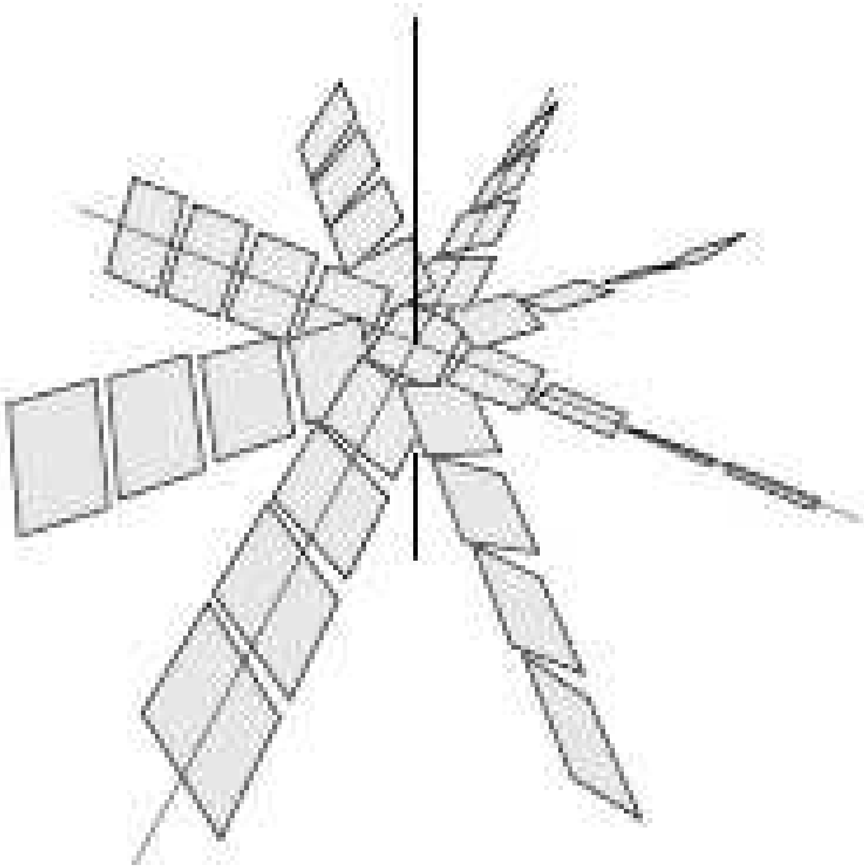

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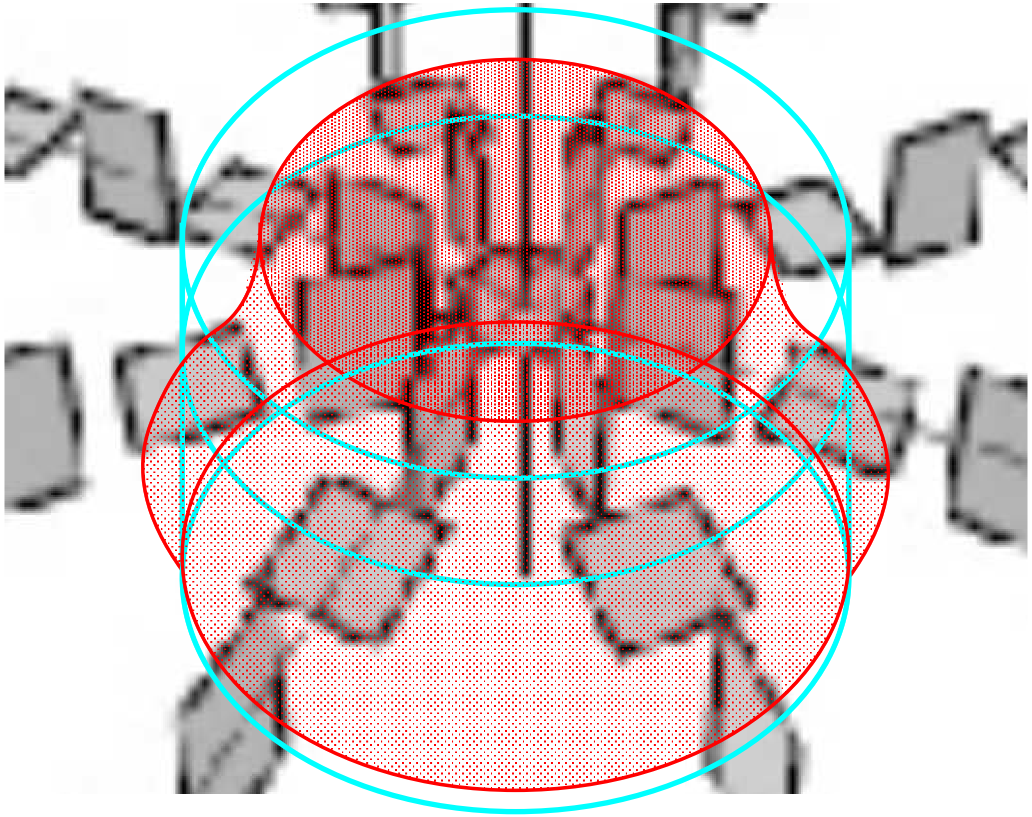


$\ker\{\cos r^2 dz + \sin r^2 d\theta\}$

§ Overtwisted disc (2).

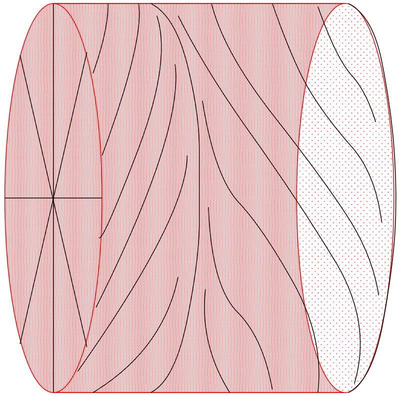
◎ In higher-dimensions

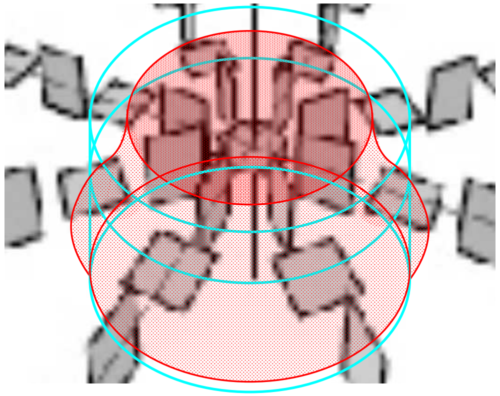
An overtwisted disc in any dimension $2n+1$

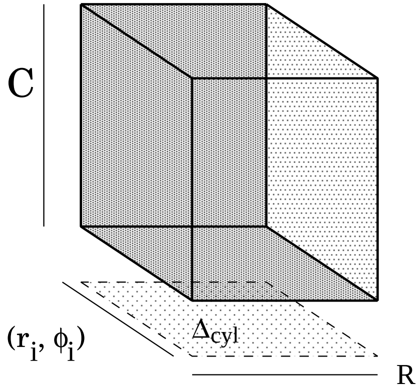
is defined as a certain piecewise smooth $2n$-disk

$$\partial(D^2 \times D^{2n-1}) \cap \{-1 \leq z \leq 1-\varepsilon\},$$

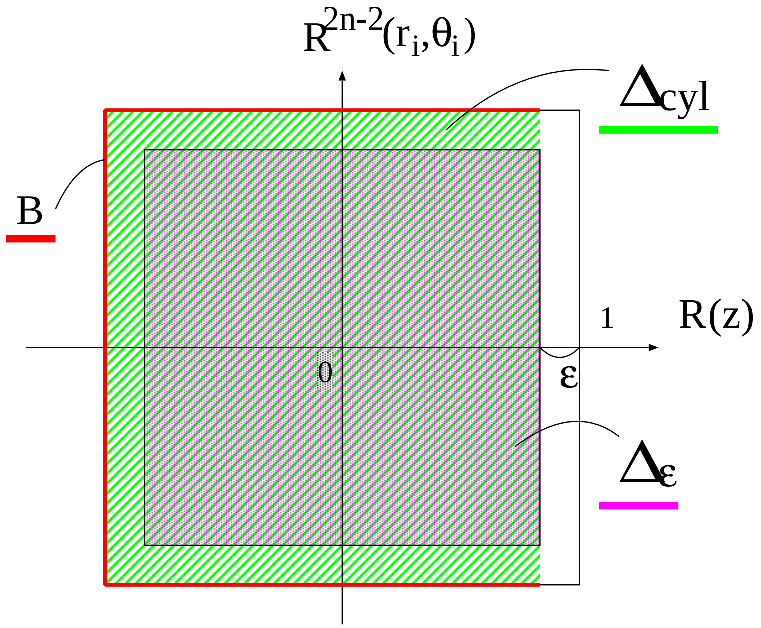
$(r, \theta, z, r_i, \theta_i) \in D^2 \times D^{2n-1}$, **with a germ of contact str.**



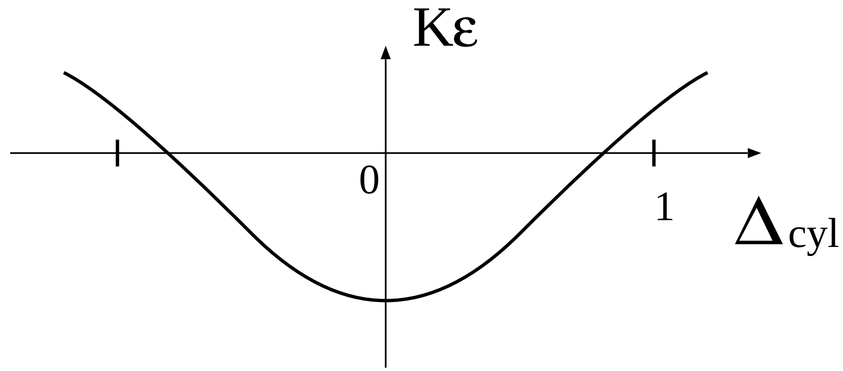




§ Overtwisted disc (3).



$K_\varepsilon : \Delta_{cyl} \to \mathbb{R}$: function



$$\Sigma_1 := \Big\{(q,v,t) \in \Delta_{\text{cyl}} \times (S^1 \times \mathbb{R}) \mid t \in S^1, v = K_\varepsilon(q)\Big\}$$

$$\subset \Big(\Delta_{\text{cyl}} \times \underline{(S^1 \times \mathbb{R})}, \ker(dz + r_i^2 d\theta_i + v dt)\Big)$$

$$\Sigma_2 := \Big\{(q,v,t) \in \Delta_{\text{cyl}} \times \mathbb{C} \mid q \in B, t \in S^1, v \in [0, K_\varepsilon(q)]\Big\}$$

$$\subset \Big(\Delta_{\text{cyl}} \times \underline{\mathbb{C}}, \ker(dz + r_i^2 d\theta_i + v dt)\Big).$$

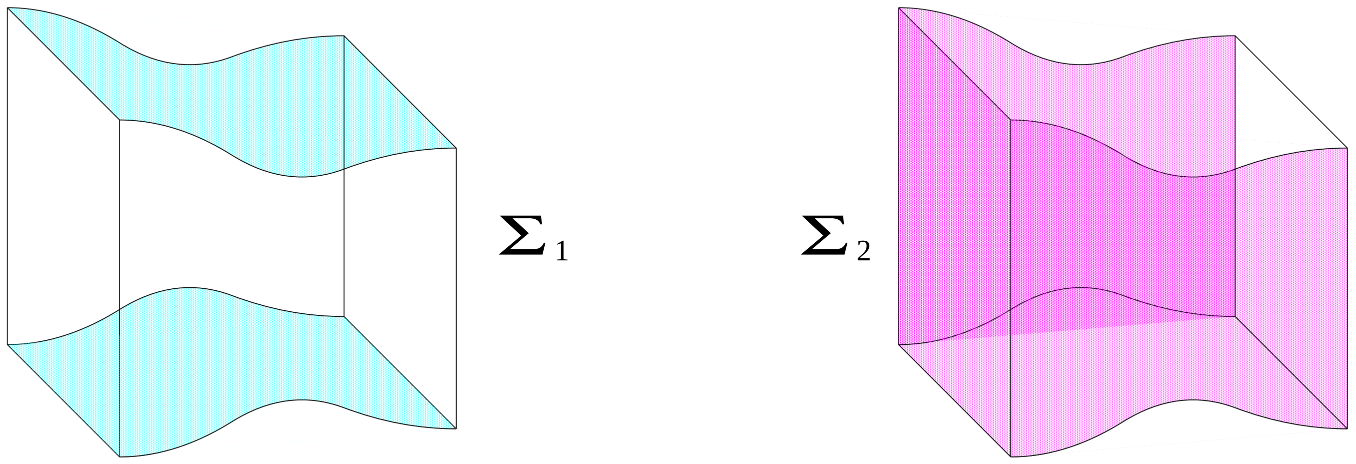
§Overtwisted disc. (4)

We obtain

a $2n$-disc

$\Sigma := \Sigma_1 \cup \Sigma_2$

with a germ of contact str.



For some constant $\varepsilon_{\text{univ}}$, if $\varepsilon < \varepsilon_{\text{univ}}$, Σ is called an overtwisted disc.

Prop. (Borman, Eliashberg, Murphy, '15)

ξ, ξ': overtwisted contact structures.

homotopic as almost contact strs $\Longrightarrow$ isotopic.

◎ **Definition**

(M, ξ): a $(2n+1)$-dimensional contact manifold,

B: an $(n-1)$-dimensional closed manifold,

$\mathcal{P}_B \subset (M, \xi)$: an $(n+1)$-dim. submanifold, $\partial \mathcal{P}_B \neq \emptyset$

is a Plastikstufe with core B

$\Longleftrightarrow$ • $\mathcal{P}_B \cong D^2 \times B$,

• each fiber $\{z\} \times B$ is isotropic (tang. to ξ) for $\forall z \in D^2$.

• On each slice $D^2 \times \{b\}$,

the line field $\xi \cap T(D^2 \times \{b\})$ generates

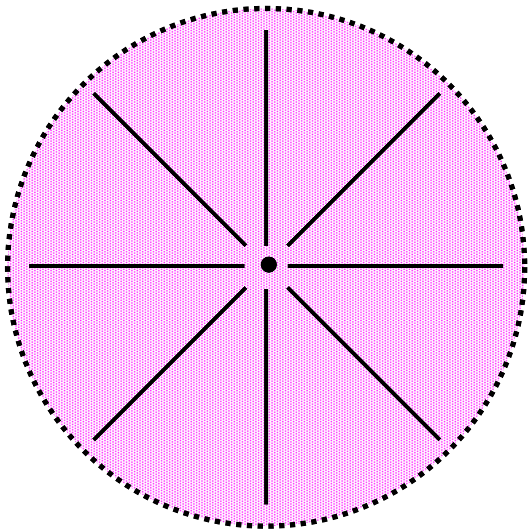
the same char. foliation as the 3-dim. OT-disc.

Rem. bLob with the trivial monodromy.

A Plastikstufe $\mathcal{P}_B \subset (M^{2n+1}, \xi)$ is small
$\Longleftrightarrow \exists D^{2n+1} \subset (M^{2n+1}, \xi)$ s.t. $D^{2n+1} \supset \mathcal{P}_B$.

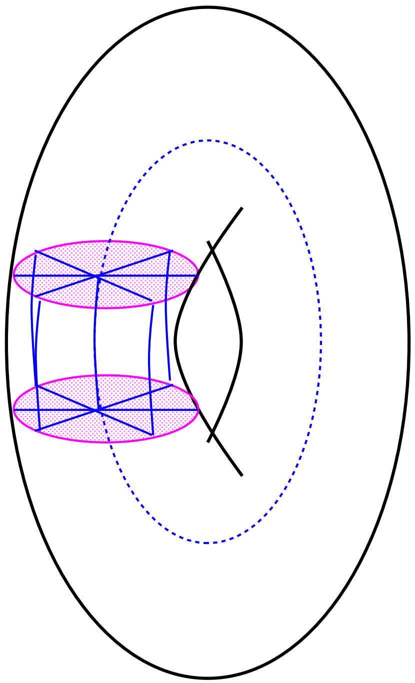
Example (plastikstufe)

3-dim.



,

5-dim.



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§Plastikstufe (3).

$\mathcal{P}_B \cong D^2 \times B$: a plastikstufe in $\subset (M^{2n+1}, \xi)$.

$(0, \varepsilon) \subset D^2 \times \{b\}$: a leaf of the characteristic foliation.

The Legendrian submanifold $(0, \varepsilon) \times B \subset \mathcal{P}_B$

is called a leaf ribbon of $\mathcal{P}_B$.

Prop. (Niederkrüger, '06, (Gromov, '86))

$\exists$(plastikstufe) $\Longrightarrow$ not weakly symplectically fillable.

◎Relative rotation class of small plastikstufe

the standard Legendrian $(0,1)\times T^{n-1}$ (**A**)

- $n=2$: $T^{n-1}=S^1$.

  For $D^2\in(M^5,\xi)$: Leg. disc (unique up to isotopy),

  int$D^2\setminus\{pt\}\cong(0,1)\times S^1$ (unique up to isotopy).

- $n\geq 3$,

  $D^n\subset(M^{2n+1},\xi)$: Leg. disc,

  $T^{n-2}\subset D^n$: triv. (unknot) emb. (unique u.t. isot.),

  $U(\cong T^{n-2}\times D^2)\subset D^n$: tubu.-nbd. of T^{n-2},

  $T^{n-2}\times($int$D^2\setminus\{pt\})\cong(0,1)\times T^{n-1}$ (unique u.t. isot.).

Let $f_{\mathrm{std}}\colon(0,1)\times T^{n-1}\to(M^{2n+1},\xi)$ denote such Leg. emb.

§Plastikstufe (5).

$\mathcal{P} \subset (M^{2n+1}, \xi)$: a small Plastikstufe with toric core,

$g_{\mathcal{P}} \colon (0,1) \times T^{n-1} \to (M, \xi)$: Leg. emb. (leaf ribbon).

The relative rotation class (Maslov class) of $g_{\mathcal{P}}$ relative to f_{std} is called the relative rotation class of $\mathcal{P}$.

In other words,

$df^{\mathbb{C}}_{\text{std}}$, $dg^{\mathbb{C}}_{\mathcal{P}} \colon T((0,1) \times T^{n-1}) \otimes \mathbb{C} \to \mathbb{C}^n$ complexifications

are (fiberwise) complex isomorphisms.

Then $\exists \varphi_{\mathcal{P}} \colon (0,1) \to \mathrm{GL}(n, \mathbb{C})$ s.t. $(df^{\mathbb{C}}_{std})_x = \varphi_{\mathcal{P}}(x) \cdot (dg^{\mathbb{C}})_x$.

The class $[\varphi_{\mathcal{P}}] \in [(0,1) \times T^{n-1}, \mathrm{GL}(n, \mathbb{C})]$

is called the relative rotation class of $\mathcal{P}_B$.

§Result.

ξ: contact structure on M^{2n+1}, $n > 1$.

Theorem. (**A.**) ξ: overtwisted $\Longleftrightarrow$

$\exists \mathcal{P}_B \subset (M, \xi)$:plastikstufe,

s.t. · small,

· $B \cong T^{n-1}$ (toric core),

· $[\phi_{\mathcal{P}_B}] = [Id]$ (trivial rotation).

◎ Generalized Lutz twists

- EP-twist, (Etnyre, Pancholi,'11?,'17)

  makes a small plastikstufe with spherical core

  and trivial rotation.

Prop. (Casals, Murphy, Presas, '15, arXiv)

ξ: overtwisted $\Longleftrightarrow$

$\exists \mathcal{P}_B \subset (M, \xi)$:plastikstufe,

s.t. · small, · spherical core, · trivial rotation.

- A-twist (A., '16, arXiv)

  makes a small plastikstufe with toric core

  and trivial rotation, and OT-disc.

§ Generalized Lutz tube.

Prop. (the standard tubu.-nbhd of a transv. curve)

$(S^1 \times \mathbb{R}^{2n}, \xi_0)$: tubu.-nbd. of $S^1 \times \{0\}$,

where $\xi_0 = \ker\{d\phi + r_1^2 d\theta_1 + \cdots + r_n^2 d\theta_n\}$.

the Lutz tube along a transv. curve in dimension 3.

$(S^1 \times \mathbb{R}^2, \zeta = \ker \omega_{\mathtt{tw}})$: tubu.-nbd. of $S^1 \times \{0\}$, where

$$\omega_{\mathtt{tw}} := (\cos r^2) d\phi + (\sin r^2) d\theta.$$

(a generalized Lutz tube along a transv. curve, (A.)).

$(S^1 \times \mathbb{R}^{2n}, \zeta = \ker \omega_{\mathtt{tw}})$: tubu.-nbd. of $S^1 \times \{0\}$, where

$$\omega_{\mathtt{tw}} := \prod_{i=1}^{n} (\cos r_i^2) d\phi + \sum_{i=1}^{n} (\sin r_i^2) d\theta_i.$$

Rem. $\ker \omega_{\mathtt{tw}}$ is a conductive confoliation.

§ Generalized Lutz tube (2).

★ **Here is an S^1-family of plastikstufes.**

$$P = \{\phi \in S^1,\ r_1 \leq \sqrt{\pi},\ r_2 = \cdots = r_n = \sqrt{\pi}\}$$

$$\cong S^1 \times (D^2 \times T^{n-1}) = S^1 \times (\textbf{plastikstufe}).$$

$$(\text{core } B = T^{n-1} \ni (\theta_2, \ldots, \theta_n))$$

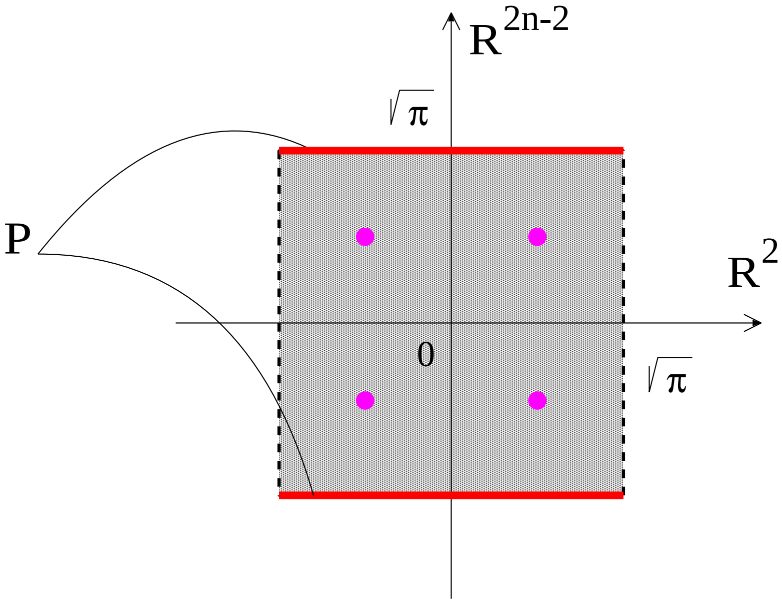
Actually,

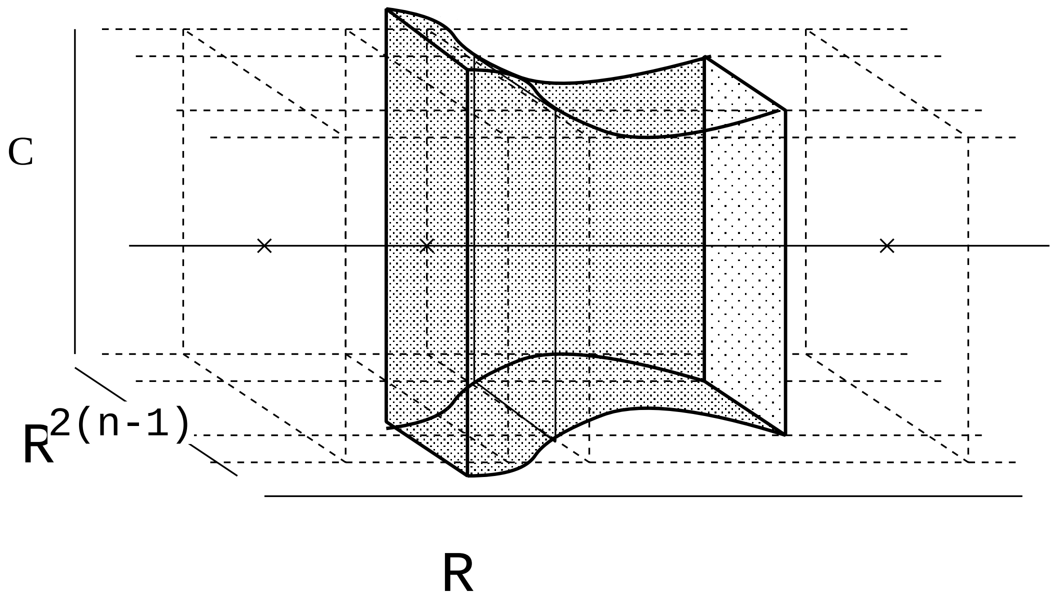
$$\omega_{\mathtt{tw}}|_{T(\{z\}\times T^{n-1})}$$
$$= \left(\prod(\cos r_i^2)d\phi + \sum(\sin r_i^2)d\theta_i\right)\Big|_{T(\{z\}\times T^{n-1})} = 0$$

$$\omega_{\mathtt{tw}}|_{T(D^2\times\{b\})}$$
$$= \left(\prod(\cos r_i^2)d\phi + \sum(\sin r_i^2)d\theta_i\right)\Big|_{T(D^2\times\{b\})} = \sin r_1^2 d\theta_1$$

§ Generalized Lutz tube (3).

◎ plastikstufe and overtwisted disc



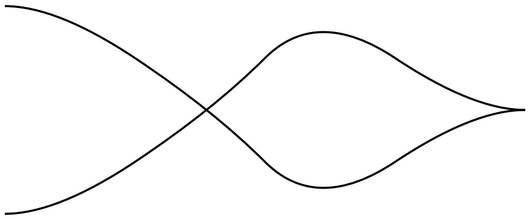


§ Sketch of the Proof — Key observations.

◎**From Loose Legendrian submanifold to OT.**

Loose Legendrian submanifold.

— contactomorphic in part to a loose chart

 $\times$ (0-section) $\subset \mathbb{R}^3 \times T^*\mathbb{R}^{n-1}$.

Prop. (Casalse, Murphy, Presas, 15, arXiv)

Trivial Legendrian sphere is loose

$\Longrightarrow \xi$ is overtwisted.

§ Sketch of the Proof — Key observations (2).

◎ **From plastikstufe to Loose Legendrian submanifold.**

<u>Theorem.</u> (**A**.)

$\exists \mathcal{P}_{T^{n-1}} \subset (M, \xi)$: plastikstufe

s.t. · small, · toric core, · trivial rotation.

$\Longrightarrow \Lambda \subset (M, \xi)$: Legendrian submfd s.t. $\Lambda \cap \mathcal{P} = \emptyset$ is loose.

§ Sketch of the Proof — Key observations (3).

Key Lemma. (**A.**) $\exists \mathcal{P}_{T^{n-1}} \subset (M, \xi)$: plastikstufe

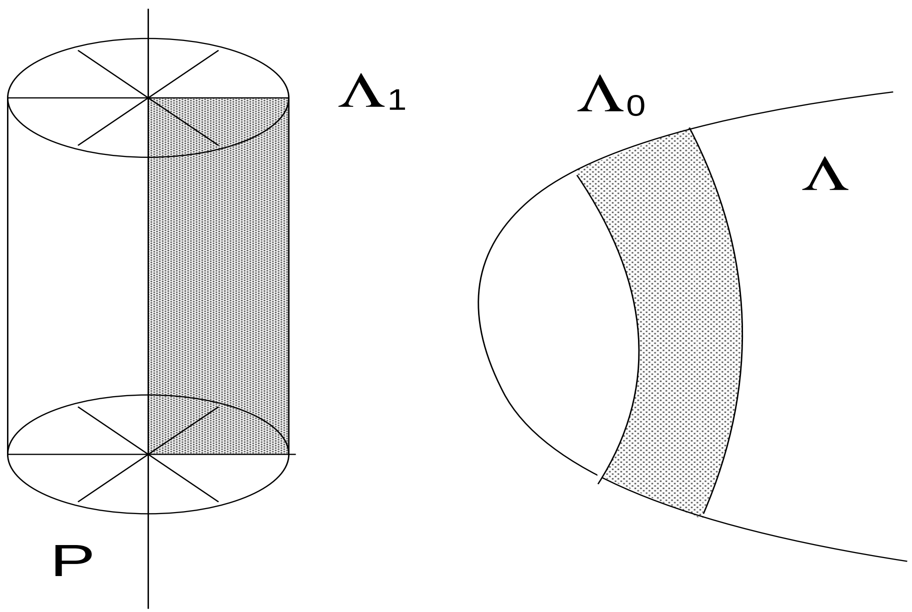
s.t. · small, · toric core, · trivial rotation.

$\Longrightarrow$ For $\forall \Lambda \subset (M, \xi)$: Leg. submfd. s.t. $\Lambda \cap \mathcal{P} = \emptyset$,

$\exists \Lambda_0 \subset \Lambda$ s.t. · $\Lambda_0 \cong (0,1) \times T^{n-1}$,

· for a leaf ribbon of $\mathcal{P}$, Λ_0 can be isotoped

to a certain good position.



§ Sketch of the Proof — Key observations (4).

Then, apply the following idea

to the family simultaneously.

Prop. (Murphy, Niederkrüger, Presas, stipsicz, 13)

(In dim. 3,) L: Leg. curve, $D_{\mathtt{ot}}$: OT-disc,

$\Longrightarrow L\#\partial D_{\mathtt{ot}}$ is the negative stabilization of L.

Summary

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