Levi-flat boundary and Levi foliation: holomorphic functions on a disk bundle

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A holomorphic disk bundle over a closed Riemann surface

- **Let Σ be a compact Riemann surface of genus ≥ 2.**
- **Uniformize $\Sigma = \mathbb{D}/\Gamma$. Extend $\Gamma \curvearrowright \mathbb{D} \subset \mathbb{CP}^1$.**

- **Diagonal action $\Gamma \curvearrowright \mathbb{D} \times \mathbb{CP}^1$ gives $X := \mathbb{D} \times \mathbb{CP}^1/\Gamma$.**
- **The first projection gives $X \to \Sigma$, a $\mathbb{CP}^1$-bundle.**

- $\Omega := \mathbb{D} \times \mathbb{D}/\Gamma$, $\Omega' := \mathbb{D} \times \mathbb{D}^*/\Gamma$ **where $\mathbb{D}^* := \mathbb{CP}^1 \setminus \overline{\mathbb{D}}$.**
- **The first projections gives $\Omega \to \Sigma$ and $\Omega' \to \Sigma$, $\mathbb{D}$-bundles.**

- $M = \partial\Omega = \partial\Omega' = \mathbb{D} \times S^1/\Gamma \to \Sigma$ **is a C^ω Levi-flat S^1-bundle.**
- M **is diffeomorphic to the unit tangent bundle of Σ.**
- **The Levi foliation is the weak stable foliation of the geodesic flow on Σ.**

Known facts

(Diederich–Ohsawa '85)

- **Ω is 1-convex. Recall that $\Omega := \mathbb{D} \times \mathbb{D}/(z, w) \sim (\gamma z, \gamma w), \gamma \in \Gamma$.**
  $\varphi := -\log \delta$, **where** $\delta := 1 - \left| \frac{w - z}{1 - \bar{z} w} \right|^2$**, is a proper smooth psh which is strictly psh except $D := \{(z, z) \mid z \in \mathbb{D}\}/\Gamma \simeq \Sigma$.**
- **Ω' is Stein. Note that $\Omega' \simeq \mathbb{D} \times \mathbb{D}/(z, w') \sim (\gamma z, \bar{\gamma} w'), \gamma \in \Gamma$.**
  $\varphi := -\log \delta'$, **where** $\delta' := 1 - \left| \frac{w - \bar{z}}{1 - z w} \right|^2$**, is a proper smooth strictly psh.**
  Ω' contains a totally real surface $D' := \{(z, \bar{z}) \mid z \in \mathbb{D}\}/\Gamma \approx \Sigma$.

(cf. E. Hopf '36)

- **Bounded holomorphic functions on Ω and Ω' are constant.**

Main Theorem

Question (asked by Ohsawa, Mitsumatsu)

Can we express holomorphic functions on Ω and Ω' explicitly? What is their growth rate?

- $\mathcal{O}(\Omega) \simeq \{f \in \mathcal{O}(\mathbb{D} \times \mathbb{D}) \mid f(z,w) = f(\gamma z, \gamma w), \gamma \in \Gamma\}$.
- **(Ohsawa)** $\sum_{\gamma \in \Gamma} (\gamma(z) - \gamma(w))^N \in \mathcal{O}(\Omega)$ **for** $N \geq 2$.

Theorem (A.)

$$I : \bigoplus_{n=0}^{\infty} H^0(\Sigma, K_\Sigma^{\otimes n}) \hookrightarrow \mathcal{O}(\Omega), \quad I' : \bigoplus_{n=0}^{\infty} \mathrm{Ker}(\Delta - \lambda_n I) \hookrightarrow \mathcal{O}(\Omega')$$

where

$$H^0(\Sigma, K_\Sigma^{\otimes n}) = \{\textbf{holomorphic } n\textbf{-differential } \psi = \psi(\tau)(d\tau)^{\otimes n} \textbf{ on } \Sigma\}$$

$\mathrm{Ker}(\Delta - \lambda_n I) = \{f : \Sigma \to \mathbb{C} \mid \Delta f = \lambda_n f\}$, Δ**: Laplacian w.r.t. Poincaré**

Theorem (A., continued)

Moreover, for any $\psi \in H^0(\Sigma, K_\Sigma^{\otimes n})$ **and** $f \in \mathrm{Ker}(\Delta - \lambda_n I)$,

$$\|I(\psi)\|_\alpha^2 = \int_\Omega |I(\psi)|^2 \delta^\alpha dV < \infty, \quad \|I'(f)\|_\alpha^2 = \int_{\Omega'} |I'(f)|^2 \delta'^\alpha dV < \infty,$$

for all $\alpha > -1$. **Here** dV **is any volume form of** $X = \mathbb{D} \times \mathbb{CP}^1/\Gamma$.

- $\Omega := \mathbb{D} \times \mathbb{D}/(z,w) \sim (\gamma z, \gamma w)$. $\delta = 1 - \left|\frac{w-z}{1-\bar{z}w}\right|^2$.
- $\Omega' \simeq \mathbb{D} \times \mathbb{D}/(z,w') \sim (\gamma z, \bar{\gamma} w')$. $\delta' = 1 - \left|\frac{w-\bar{z}}{1-zw}\right|^2$.

(E. Hopf '36, L. Garnett '83)

- **For** $f \in \mathcal{O}(\Omega)$ **or** $\mathcal{O}(\Omega')$, $\|f\|_\alpha^2 = o\left(\frac{1}{\alpha+1}\right)$ **as** $\alpha \searrow -1$ **(i.e.** f **belongs to the Hardy space / has** L^2 **boundary value)** $\implies$ f **is constant.**

Outline of Proof

$I : \bigoplus_{n=0}^{\infty} H^0(\Sigma, K_\Sigma^{\otimes n}) \hookrightarrow \mathcal{O}(\Omega)$ **is given by, for** $\psi \in H^0(\Sigma, K_\Sigma^{\otimes n})$**,** $n \geq 1$**,**

$$I(\psi)(z,w) = \int_z^w \frac{1}{B(n,n)} \left(\frac{(w-\tau)(\tau-z)}{(w-z)d\tau} \right)^{\otimes(n-1)} \psi(\tau)(d\tau)^{\otimes n}$$

where $\psi = \psi(\tau)(d\tau)^{\otimes n}$ **on** $\mathbb{D}_\tau$ **and** $B(p,q)$ **is the beta function.**

$I' : \bigoplus_{n=0}^{\infty} \mathrm{Ker}(\Delta - \lambda_n I) \hookrightarrow \mathcal{O}(\Omega')$ **is given by the analytic continuation of given** $f : \Sigma \to \mathbb{C}, \Delta f = \lambda_n f$ **as a function on** $\mathbb{D} \simeq \{(z, \bar{z}) \mid z \in \mathbb{D}\}/\Gamma$ **to** $\mathbb{D} \times \mathbb{D}$**. It follows that** f **actually extends to entire** $\mathbb{D} \times \mathbb{D}$ **from the method to show the integrability of** $I(\psi)$ **above.**

Outline of proof for the integrability

The idea of the formula defining I comes from

$$K_\Sigma \simeq T^*_\Sigma \simeq T^*_D \simeq N^*_{D/\Omega}, \quad D = \{(z,z) \mid z \in \mathbb{D}\}/\Gamma \subset \Omega,$$

which implies $\psi \in H^0(\Sigma, K_\Sigma^{\otimes n})$ is identified with a n-jet of a holomorphic function along $D \subset \Omega$. $I(\psi)$ gives the extension of ψ which has the smallest $\|\cdot\|_\alpha$ norm.

Step 1. **We use a non-holomorphic coordinate of $\mathbb{D}_z \times \mathbb{D}_w$, (z,t) given by $t = (w-z)(1-\overline{z}w)^{-1}$. $f = f(z,w) \in \mathcal{O}(\Omega)^\Gamma$ is expanded as $f = \sum_{n=0}^\infty f_n(z)t^n$ and $\{f_n\}$ satisfy**

$$\frac{\partial f_n}{\partial \overline{z}} + \frac{nz}{1-|z|^2} f_n + \frac{n-1}{1-|z|^2} f_{n-1} = 0.$$

Put $\varphi_n := f_n(z)\left(\frac{\sqrt{2}dz}{1-|z|^2}\right)^{\otimes n} \in C^{(0,0)}(\Sigma, K_\Sigma^n)$. Then $\{\varphi_n\}$ satisfy

$$\overline{\partial}\varphi_0 = 0, \quad \overline{\partial}\varphi_n = -\frac{n-1}{\sqrt{2}}\varphi_{n-1} \otimes \omega \ (n \geq 1)$$

where $\omega = 2dz \otimes d\overline{z}/(1-|z|^2)^2$.

Outline of proof for the integrability — continued

Step 2.

Let $\psi \in H^0(\Sigma, K_\Sigma^{\otimes N})$. Put $\varphi_n := 0$ for $n < N$ and $\varphi_N := \psi$. We pick the L^2 minimal solution to

$$\overline{\partial}\varphi_n = -\frac{n-1}{\sqrt{2}}\varphi_{n-1} \otimes \omega$$

inductively and determine φ_n for $n > N$. The spectral decomposition of the complex laplacian tells us the L^2 minimal solutions are

$$\begin{aligned}\varphi_{N+m} &= \overline{\partial}^*_{N+m} G^{(1)}_{N+m}\left(-\frac{N+m-1}{\sqrt{2}}\varphi_{N+m-1} \otimes \omega\right)\\ &= -\frac{\sqrt{2}(N+m-1)}{m(2N+m-1)}\overline{\partial}^*_{N+m}(\varphi_{N+m-1} \otimes \omega)\end{aligned}$$

where $\overline{\partial}^*_n$ is the formal adjoint of $\overline{\partial} : C^{(0,0)}(\Sigma, K_\Sigma^{\otimes n}) \to C^{(0,1)}(\Sigma, K_\Sigma^{\otimes n})$ and $G^{(1)}_n$ is the Green operator on $C^{(0,1)}(\Sigma, K_\Sigma^{\otimes n})$.

Outline of proof for the integrability — continued^2

Step 3.

The convergence of $f = \sum_{n=0}^{\infty} f_n(z) t^n$ in $L^2(\Omega)$, $\varphi_n = f_n(z) \left(\frac{\sqrt{2}dz}{1-|z|^2}\right)^{\otimes n}$, follows from

$$
\begin{aligned}
\|f\|_0^2 &= \pi \sum_{n=0}^{\infty} \frac{\|\varphi_n\|^2}{n+1} \\
&= \pi \sum_{m=0}^{\infty} \frac{\|\varphi_{N+m}\|^2}{N+m+1} \\
&= \frac{1}{B(N,N)} \|\psi\|^2 \sum_{m=0}^{\infty} \frac{\{(N+m-1)!\}^2}{m!(2N+m-1)!(N+m+1)} < \infty.
\end{aligned}
$$

Similar computation shows $\|f\|_\alpha < \infty$ for any $\alpha < 1$.

Outline of proof for the integrability — continued[3]

Step 4.

Want to show

$$\sum_{n=0}^{\infty} f_n(z)t^n = \int_z^w \frac{1}{B(N,N)} \left(\frac{(w-\tau)(\tau-z)}{(w-z)d\tau}\right)^{\otimes(N-1)} \psi(\tau)(d\tau)^{\otimes N}.$$

Enough to show the desired equality on $\{0\} \times \mathbb{D}$**.**

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n(0)t^n &= \frac{(2N-1)!}{(N-1)!} \sum_{m=0}^{\infty} \frac{(N+m-1)!}{(2N+m-1)!} \frac{1}{m!} \frac{\partial^m \psi}{\partial z^m}(0) t^{N+m} \\
&= \frac{(2N-1)!}{(N-1)!} t^N \int_0^1 dt_N \ldots \int_0^{t_3} dt_2 \int_0^{t_2} t_1^{N-1} \psi(tt_1) dt_1 \\
&= \frac{(2N-1)!}{(N-1)!} t^N \int_0^1 \frac{t_1^{N-1}(1-t_1)^{N-1}}{(N-1)!} \psi(tt_1) dt_1 \\
&= \int_0^t \frac{1}{B(N,N)} \left(\frac{(t-\tau)\tau}{t}\right)^{(N-1)} \psi(\tau) d\tau. \qquad \square
\end{aligned}$$