不定値計量と接触構造の双対性と特異性

Duality-singularity of indefinite metrics and contact structures

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A talk based on joint-works with

Y. Machida  and M. Takahashi 

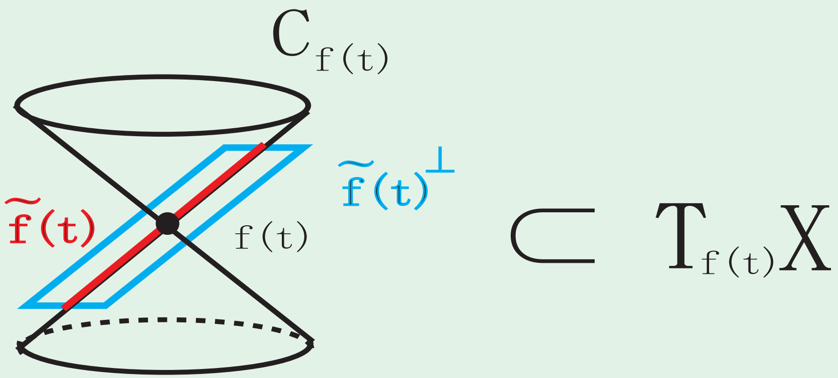
Let (X, g) be a C^∞ Lorentz 3-manifold (of signature $(1, 2)$).

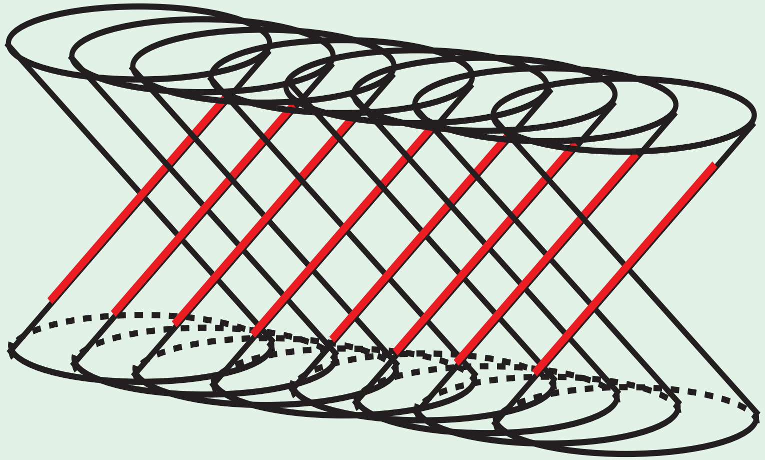
Example (Minkowskii space):
$X = \mathbf{R}^3, \quad g : ds^2 = dx_1^2 - dx_2^2 - dx_3^2.$

Definition. A C^∞ immersion $f : U(\subset \mathbf{R}^2) \to (X, g)$ is called null (or lightlike) if the pull-back metric f^*g is degenerate everywhere on U.

Let $C := \{v \in TX \mid g(v, v) = 0\}$ be the null quadratic cone field associated to the indefinite metric g.

It is easy to see that an immersion $f : U \to X$ is null if and only if $f_*(T_t U)$ is tangent to the cone $C_{f(t)}$ for any $t \in U$.
Then U is foliated by null curves.
(A curve is called null if its velocity vectors belong to $C \subset TX$.)





Let $Z := PC = \{(x, \ell) \mid x \in X, \ell \text{ is a null line in } T_xX\} \subset P(TX)$, the space of null directions, $\dim(Z) = 4$.
Denote by $\pi_X : Z \to X$ the natural projection, $\pi_X(x, \ell) = x$.

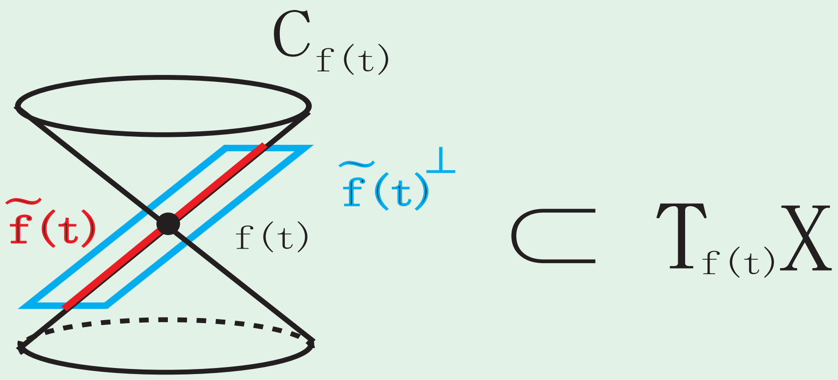
Lemma. Let $f : U \to X$ be a null immersion. Then there exists a unique C^∞ map $\widetilde{f} : U \to Z$ such that $\pi_X \widetilde{f} = f$ and that $f_*(T_tU) = \widetilde{f}(t)^\perp$, for any $t \in U$.

$\widetilde{f}(t)^\perp := \{v \in T_{f(t)}X \mid g(v, u) = 0, \text{ for any } u \in \widetilde{f}(t)\}$.

$$\begin{array}{ccccc} & & Z & \hookrightarrow & PT^*X \\ & \widetilde{f} \nearrow & \downarrow \pi_X & \swarrow & \\ U & \xrightarrow{f} & X & & \end{array}$$

Definition. A (possibly non-immersive) C^∞ map-germ $f : (\mathbf{R}^2, p) \to X$ is called a null frontal surface or a lightlike frontal surface if there exists a C^∞ map-germ $\widetilde{f} : (\mathbf{R}^2, p) \to Z$ such that $\pi_X \widetilde{f} = f$ and that $f_*(T_t\mathbf{R}^2) \subset \widetilde{f}(t)^\perp$, for any $t \in \mathbf{R}^2$ nearby p.

$$\widetilde{f}(t)^\perp := \{v \in T_{f(t)}X \mid g(v, u) = 0, \text{ for any } u \in \widetilde{f}(t)\}.$$

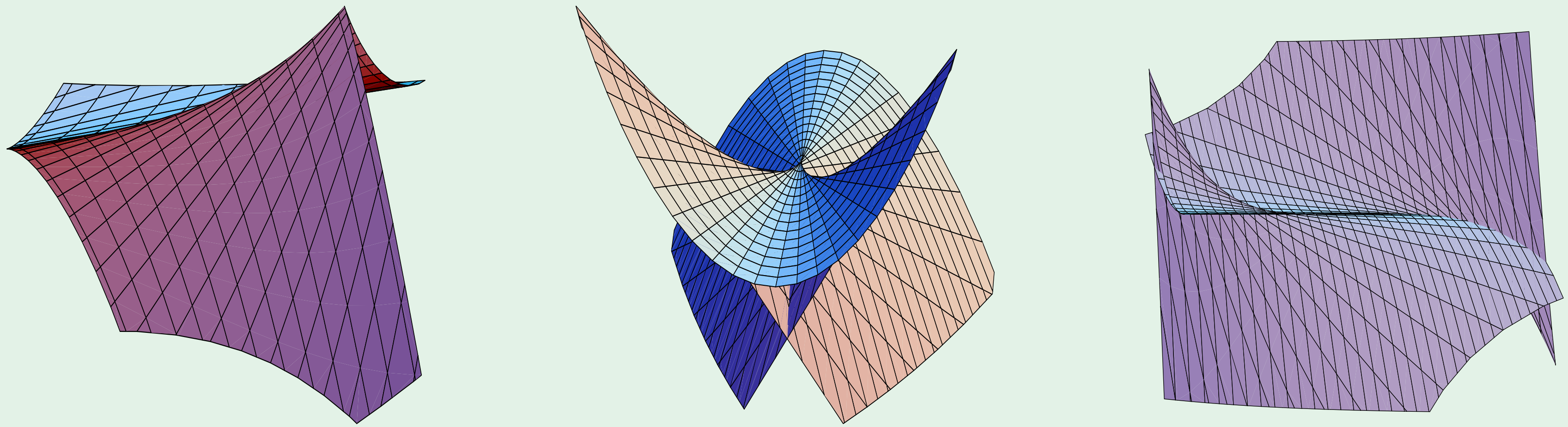


Problem. Classify "generic" singularities of null frontal surfaces up to local diffeomorphisms.

— Minkowskii case: Chino-Izumiya 2010 (lightlike developables).
— $O(2,3)$-homogeneous case: Machida-Takahashi-I 2011 (tangent surfaces of directed null curves).

We show that any null frontal surface is a null tangent surface of a directed null curves (in a wider sense) and we give the generic classification of singularities of null frontal surfaces for general Lorentz 3-manifolds, moreover for arbitrary non-degenerate (strictly convex) cone fields on 3-manifolds.

Solution. The list of generic singularities consists of three local diffeomorphism classes, cuspidal edge (CE), swallow-tail (SW) and Shcherbak surface (SB).



Cuspidal Edge SWallowtail ShcherBak surface

We observe the "stability" or, in other words, "robustness" of the classification of singularities.

Query. What are the geometric structures behind, which dominate the appearance of singularities of null frontals.

【 Pseudo-product Engel structure 】

Define a distribution

$$E = \bigcup_{(x,\ell)\in Z} (\pi_X)_*^{-1}(\ell) \subset TZ,$$

over Z, where $\pi_{X*} : TZ \to TX$ is the differential map of π_X.

Then E is of rank 2 with the growth $(2, 3, 4)$, i.e. $\mathcal{E}^2 := \mathcal{E} + [\mathcal{E}, \mathcal{E}]$ is generated by a subbunde E^2 of rank 3 and $\mathcal{E}^2 + [\mathcal{E}, \mathcal{E}^2] = \mathcal{TZ}$, in other words, E is an Engel structure on the 4-manifold Z.

Define other distributions

$$E_1 := \mathrm{Ker}(\pi_{X*}) \subset E \subset TZ, \ \ E_2 := \mathrm{ch}(E^2) \subset E \subset TZ,$$

Cauchy characteristic of E^2.

Then both E_1 and E_2 are of rank 1, and we have the decomposition

$$E = E_1 \oplus E_2,$$

a pseudo-product Engel structure (in the sense of Noboru Tanaka). Moreover we have

$$\mathcal{E} + [\mathcal{E}_1, \mathcal{E}] = \mathcal{E}^2, \quad \mathcal{E}^2 + [\mathcal{E}_1, \mathcal{E}^2] = \mathcal{TZ}, \quad [\mathcal{E}_2, \mathcal{E}^2] \subset \mathcal{E}^2.$$

Let Y denote the leaf space of E_2, which is regarded as the space of null geodesics.

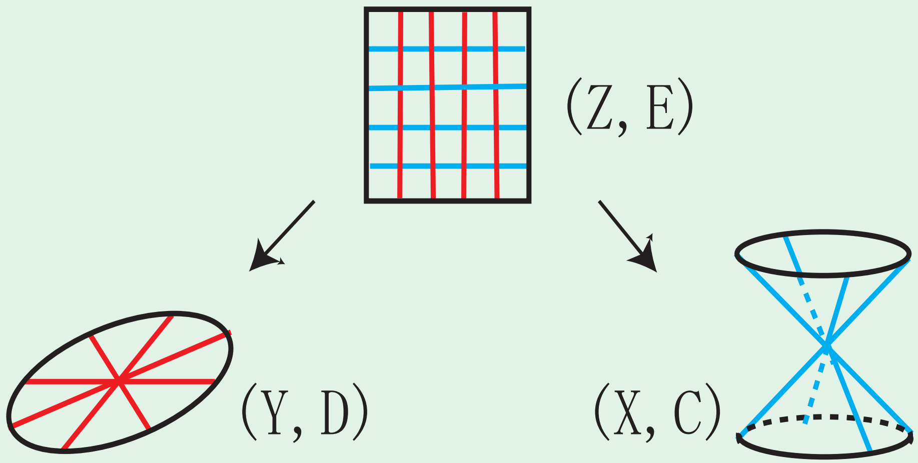
Denote by $\pi_Y : Z \to Y$ be the natural projection.

Locally we have the double fibration

$$Y^3 \xleftarrow{\pi_Y} Z^4 \xrightarrow{\pi_X} X^3.$$

The distribution E^2 on Z of rank 3 descends by π_Y to a contact structure D on Y:

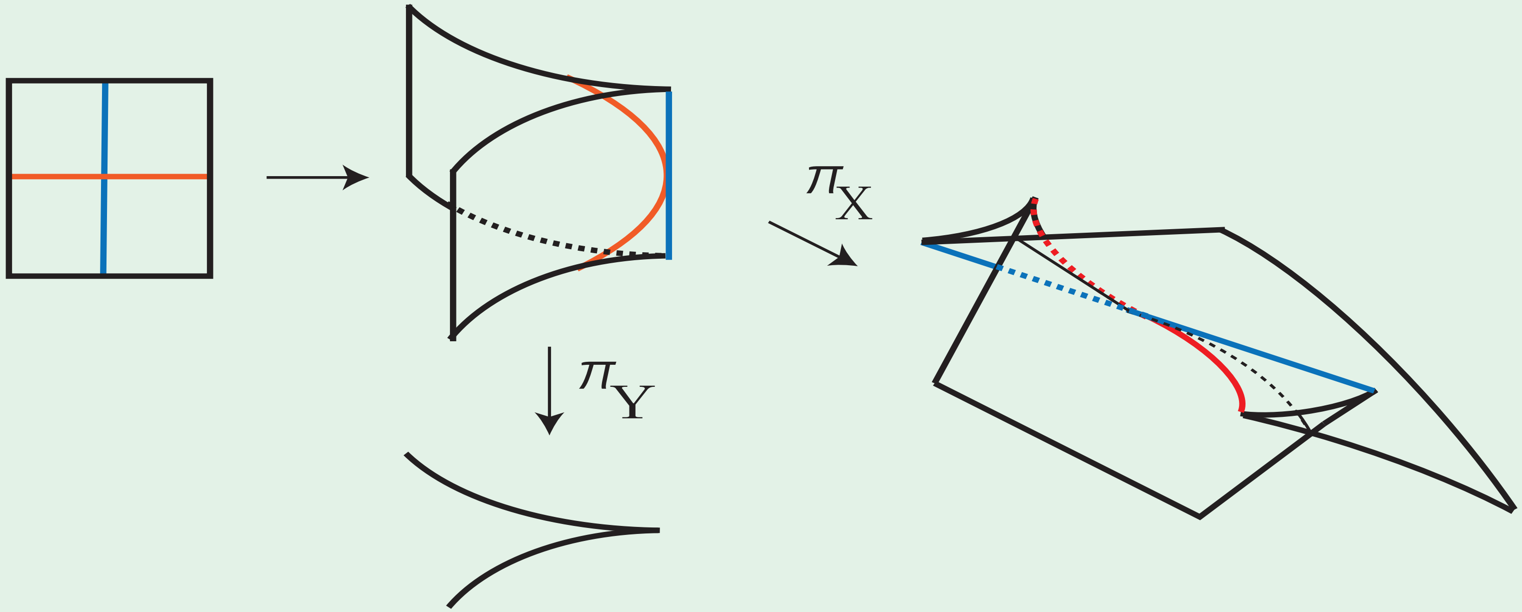
$$(\pi_{Y*})(E^2) = D, \quad E^2 = (\pi_{Y*})^{-1}(D).$$



【 Singularities of null frontal surfaces 】

Lemma. $f : (\mathbf{R}^2, p) \to X$ is a null frontal if and only if there exists an E^2-integral lift $\widetilde{f} : (\mathbf{R}^2, p) \to Z$ of f. $(\widetilde{f})_*(T_t\mathbf{R}^2) \subset (E^2)_{\widetilde{f}(t)}$, for any $t \in \mathbf{R}^2$ nearby p.
Then $\pi_Y \widetilde{f}$ is D-integral and, therefore, of rank ≤ 1.

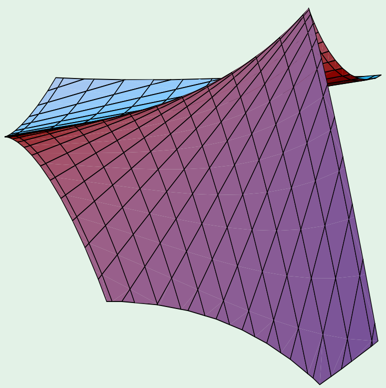
Thus $\widetilde{f}$ collapses by π_Y to a D-integral curve, in other words, to a "Legendre curve", and $\widetilde{f}$ is foliated by π_Y-fibres. Therefore f is ruled by a "Legendre family" of null geodesics.
The singular locus of $f = \pi_X \widetilde{f}$ consist of an E-integral curve γ and the π_Y-fibres of singular points of the Legendre curve.



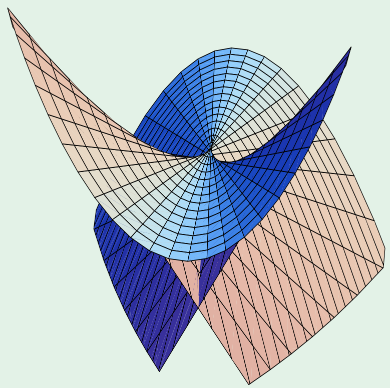
Thus the null frontal f is regarded as the tangent surface to a (directed) null curve $\pi_X \gamma$.

Avoiding very degenerate cases, we start with a germ of E-integral curve $\gamma : (\mathbf{R}, t_0) \to Z$ so that $\pi_Y^{-1}\pi_Y\gamma$ is right equivalent to $\widetilde{f}$, and f is right equivalent to $\pi_X\pi_Y^{-1}\pi_Y\gamma$.

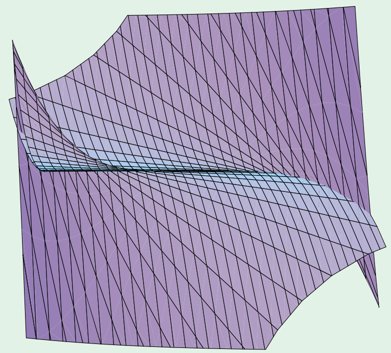
Theorem 1. (Machda-Takahashi-I)
For a generic E-integral curve $\gamma : I \to Z$, the induced null frontal $\pi_X \pi_Y^{-1} \pi_Y \gamma$ is diffeomorphic (right-left equivalent) along γ to cuspidal edge, swallowtail or Shcherbak surface. The same classification result holds for arbitrary non-degenerate (strictly convex) cone structure $C \subset TX$.



Cuspidal Edge



SWallowtail



ShcherBak surface

$$(Z,\ E = E_1 \oplus E_2)$$

pseudo-product Engel structure

3rd order ODE ↙ ↘

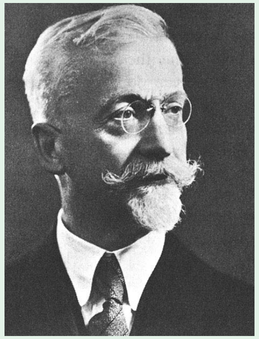
$$(Y, D)$$

projective contact structure

$$(X, C)$$

non-degenerate (strictly convex) cone structure

Classification of geometric structures, contact geometry of 3rd order ODE, by E. Cartan, S.-S. Chern(1940), N. Tanaka,



"Wünschmann invariant" $= 0 \Longleftrightarrow C$: metric cone.

There are works by H. Sato, T. Ozawa, A. Yamada-Yoshikawa, Simonetta Frittelli, C. Kozameh, E.T. Newman, and so forth.

Also there are relations with the theory of Lie contact structures, Lie tensor metric structures (Sato, Yamaguchi, Miyaoka), Grassmannian structures, (Machida, Sato), CR geometry, "sub-semi-Riemannian geometry", etc..

(Please refer the next afternoon lecture by Y. Machida.)

Some details of the classification.

Let $C \subset TX$ be a non-degenerate cone structure of X.

Then $\exists$local.coord. x_1, x_2, x_3, θ of Z such that

$\pi_X : (x_1, x_2, x_3, \theta) \mapsto (x_1, x_2, x_3)$,

$E_1 = \left\langle \frac{\partial}{\partial \theta} \right\rangle, E_2 = \left\langle \frac{\partial}{\partial x_1} + \theta \frac{\partial}{\partial x_2} + a(x,\theta) \frac{\partial}{\partial x_3} + e(x,\theta) \frac{\partial}{\partial \theta} \right\rangle$,

(e is determined from a, C non-degenerate $\Leftrightarrow a_{\theta\theta} \neq 0$.

If $a = \frac{1}{2}\theta^2$, then $e = 0$ and C is flat.)

Theorem 2. Let $\gamma : (\mathbf{R}, t_0) \to Z$ be an E-integral curve, $f = \pi_X \pi_Y^{-1} \pi_Y \gamma : (\mathbf{R}^2, (t_0, 0)) \to X$ the null frontal generated by $\pi_Y \gamma$. Set $\varphi(t) := \theta'(t) - (e \circ \gamma)(t) x_1'(t)$. Then

$f \sim CE \Longleftrightarrow x_1'(t_0) \neq 0, \varphi(t_0) \neq 0$

$f \sim SW \Longleftrightarrow x_1'(t_0) = 0, \varphi(t_0) \neq 0, x_1''(t_0) \neq 0$

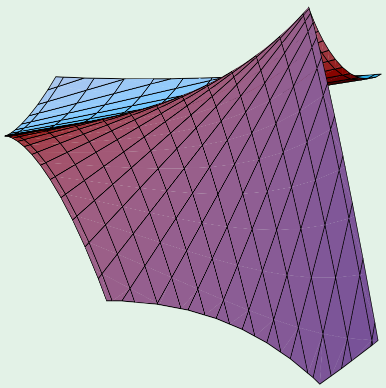
$f \sim SB \Longleftrightarrow x_1'(t_0) \neq 0, \varphi(t_0) = 0, \varphi'(t_0) \neq 0$

Here $\gamma(t) = (x_1(t), x_2(t), x_3(t), \theta(t))$.

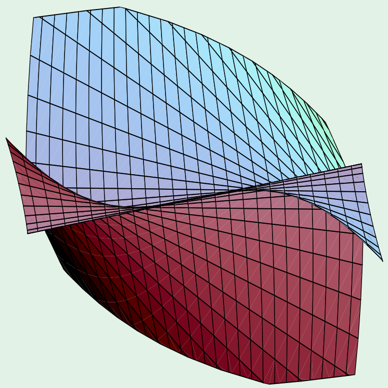
Note that $\pi_X \gamma$ is a null geodesic $\Longleftrightarrow \varphi(t) \equiv 0$.

The dual objects to null frontals are given by tangent surfaces $\pi_Y \pi_X^{-1} \pi_X \gamma$ of Legendre curves $\pi_Y \gamma$ ruled by tangential "Legendre geodesics" (Legendre lines) $\pi_Y \pi_X^{-1}(x), (x \in X)$.

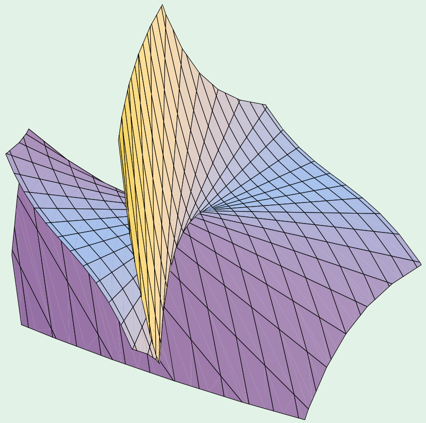
Theorem 3. For a generic E-integral curve $\gamma : I \to Z$, $\pi_Y \pi_X^{-1} \pi_X \gamma$ is diffeomorphic to cuspidal edge (CE), Mond surface (MD), or generic folded pleat (GFP).



cuspidal edge



Mond surface



generic folded pleat

$\exists$local coordinates x, y, p, q of Z such that

$\pi_Y : (x, y, p, q) \mapsto (x, y, p)$

$$E_1 = \left\langle \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + f(x, y, p, q)\frac{\partial}{\partial q} \right\rangle, \quad E_2 = \left\langle \frac{\partial}{\partial q} \right\rangle.$$

For an E-integral curve $\gamma(t) = (x(t), y(t), p(t), q(t))$, we put $\psi(t) := q'(t) - (f \circ \gamma)(t)x'(t)$.

Theorem 4. $\pi_Y \pi_X^{-1} \pi_X \gamma$ is diffeomorphic at $(t_0, 0)$ to
cuspidal edge (CE) $\Longleftrightarrow x'(t_0) \neq 0, \psi(t_0) \neq 0$,
Mond surface (MD) $\Longleftrightarrow x'(t_0) \neq 0, \psi(t_0) = 0, \psi'(t_0) \neq 0$,
folded pleat (FP) $\Longleftrightarrow x'(t_0) = 0, \psi(t_0) \neq 0, x''(t_0) \neq 0$.

Note that $\pi_Y \gamma$ is a solution of 3rd order ODE $q' = f(x, y, p, q)$ $\Longleftrightarrow \psi(t) \equiv 0$.

Thanks to Engel integral curves γ ("dancing on the heaven"), we have got the "asymmetric duality" of singularities of tangent surfaces.

Y	Z	X
$\pi_Y \pi_X^{-1} \pi_X \gamma$	γ	$\pi_X \pi_Y^{-1} \pi_Y \gamma$
CE	(I) non-tangent to E_1 non-tangent to E_2	CE
MD	(II) simply tangent to E_1 non-tangent to E_2	SW
FP	(III) non-tangent to E_1 simply tangent to E_2	SB

How about the higher dimensional cases ?

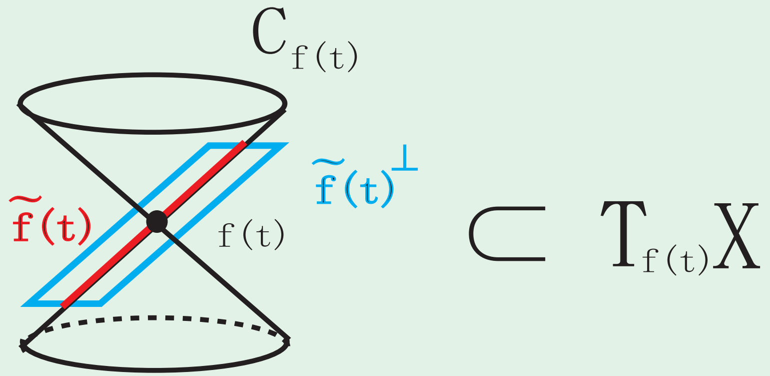
Let X be an $(n+2)$-dim. C^∞ manifold with a conformal class $[g]$ of an indefinite metric g of signature (r, s), $r + s = n + 2$, e.g. $(r, s) = (1, n + 1)$ Lorentz, $(r, s) = (2, 2)$ neutral, etc..

Let $C := \{v \in TX \mid g(v, v) = 0\}$ be the null quadratic cone field.

Set $Z := PC = \{(x, \ell) \mid x \in X, \ell \text{ is a null line in } T_xX\} \subset P(TX)$, the space of null directions, $\dim(Z) = 2n + 2$.

Denote by $\pi_X : Z \to X$ the natural projection, $\pi_X(x, \ell) = x$.

Definition. A C^∞ map-germ $f : (\mathbf{R}^{n+1}, p) \to X$ is called a null frontal (hypersurface) or a lightlike frontal (hypersurface) if there exists a C^∞ map-germ $\widetilde{f} : (\mathbf{R}^{n+1}, p) \to Z$ such that $\pi_X \widetilde{f} = f$ and that $f_*(T_t\mathbf{R}^{n+1}) \subset \widetilde{f}(t)^\perp$, for any $t \in \mathbf{R}^{n+1}$ nearby p.



Note that $\widetilde{f}(t)^{\perp}$ is a lightlike hyperplane.
An immersion to Z is called null if the pull-back of the metric g is degenerate everywhere. Then null immersions are null frontals.

For example, null hyperplanes and null cones, in the flat case, are (non-generic) null frontal hypersurfaces.

Problem. Understand the geometry of null frontal hypersurfaces. Moreover classify "generic" singularities of null frontal hypersurfaces of $(X^{n+2}, [g])$.

To try to understand the geometry well, we observe a duality between indefinite metrics and contact structures.

Define a distribution $E = \bigcup_{(x,\ell)\in Z} (\pi_X)_*^{-1}(\ell) \subset TZ$, which is of rank $n+1$ and with the growth $(n+1, 2n+1, 2n+2)$:
$\mathcal{E}^2 := \mathcal{E} + [\mathcal{E}, \mathcal{E}]$ is generated by a subbunde E^2 of rank $2n+1$ and $\mathcal{E}^2 + [\mathcal{E}, \mathcal{E}^2] = \mathcal{TZ}$.

Let $E_1 := \mathrm{Ker}(\pi_{X*})$. Then E_1 is an integrable subbundle of E of rank n.

Moreover we see that the distribution E^2 has the Cauchy characteristic $E_2 := ch(E^2)$, which is a subbundle of E of rank 1.

Then we have a pseudo-product structure

$$E = E_1 \oplus E_2, \quad \mathrm{rank}(E_1) = n, \ \mathrm{rank}(E_2) = 1,$$

$(E_1, E_2$, integrable) over Z (in the sense of Noboru Tanaka).

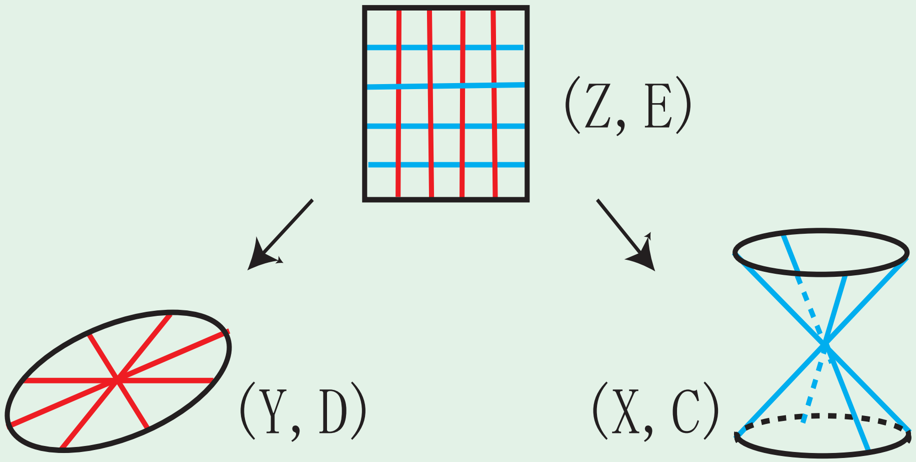
Let Y denote the leaf space of E_2, which is regarded as the space of null geodesics. $\dim(Y) = 2n + 1$. [null geodesic = (right equivalent to) null curve with paralell velocities.]

Denote by $\pi_Y : Z \to Y$ be the natural projection.

Then locally we have a double fibration

$$Y^{2n+1} \xleftarrow{\pi_Y} Z^{2n+2} \xrightarrow{\pi_X} X^{n+2}.$$

The distribution E^2 on Z of rank $2n + 1$ descends by π_Y to a contact structure D on Y. $\pi_{Y*}(E^2) = D$, $(\pi_{Y*})^{-1}(D) = E^2$.



【 General structure of null frontal hypersurfaces 】

Proposition. $f : (\mathbf{R}^{n+1}, p) \to X^{n+2}$ is a null frontal if and only if there exists an E^2-integral lift $\widetilde{f} : (\mathbf{R}^{n+1}, p) \to Z^{2n+2}$ of f. i.e. $\widetilde{f}_*(T_t\mathbf{R}^{n+1}) \subset (E^2)_{\widetilde{f}(t)}$, for any $t \in \mathbf{R}^{n+1}$ nearby p.

Then $\pi_Y \widetilde{f}$ is D-integral and of rank $\leq n$.

Then $\widetilde{f}$ is foliated by π_Y-fibres in Z.

Thus f is ruled by a "Legendre family" of null geodesics in X.

【 From Legendre maps to null frontal hypersurfaces 】

Let $\beta : (\mathbf{R}^n, p) \to Y^{2n+1}$ be a D-integral map-germ and $\beta : U \to Y$ a representative of β, written by the same letter. Then we set

$$U \times_Y Z (= \beta^{-1} Z = \pi_Y^{-1} U) := \{(u, z) \in U \times Z \mid \beta(u) = \pi_Y(z)\},$$

which is an $(n+1)$-dimensional manifolds with the fibre product:

$$\begin{array}{ccc}
\beta^{-1} Z & \xrightarrow{\pi_Y^{-1}\beta} & Z \\
\beta^{-1}\pi_Y \downarrow & \square & \downarrow \pi_Y \\
\mathbf{R}^n \supset U & \xrightarrow{\beta} & Y
\end{array}$$

Proposition. Let $\beta : (\mathbf{R}^n, p) \to Y^{2n+1}$ be a D-integral map-germ. Then the induced map $\pi_X \pi_Y^{-1} \beta : \beta^{-1} Z \to X^{n+2}$ is a null frontal hypersurface,

i.e., for any $(u_0, z_0) \in \beta^{-1} Z$, there exists a lift $\widetilde{\pi_X \pi_Y^{-1} \beta} : (\beta^{-1} Z, (u_0, z_0)) \to Z$ of $\pi_X \pi_Y^{-1} \beta$ such that

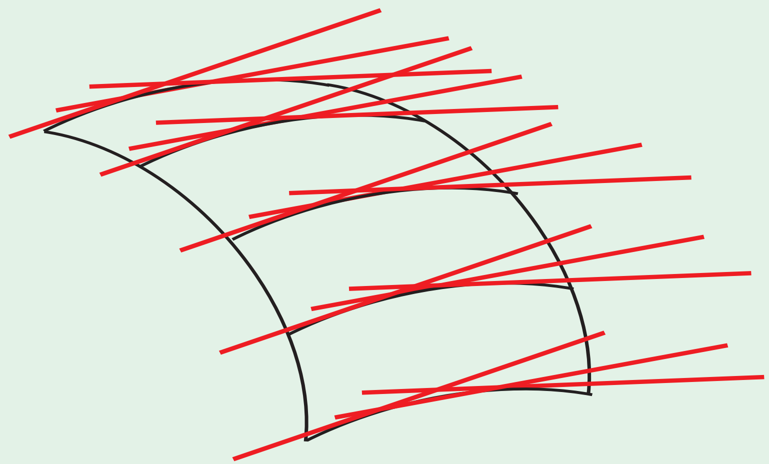
$$(\pi_X \pi_Y^{-1} \beta)_* (T_{(u,z)}(\beta^{-1} Z)) \subseteq (\widetilde{\pi_X \pi_Y^{-1} \beta})(u, z)^{\perp},$$

for any (u, z) nearby $(u_0, z_0) \in \beta^{-1} Z$.

【 Null family of null curves and their tangent 3-folds 】

$n = 2, \dim(X) = 4, \dim(Z) = 6.$

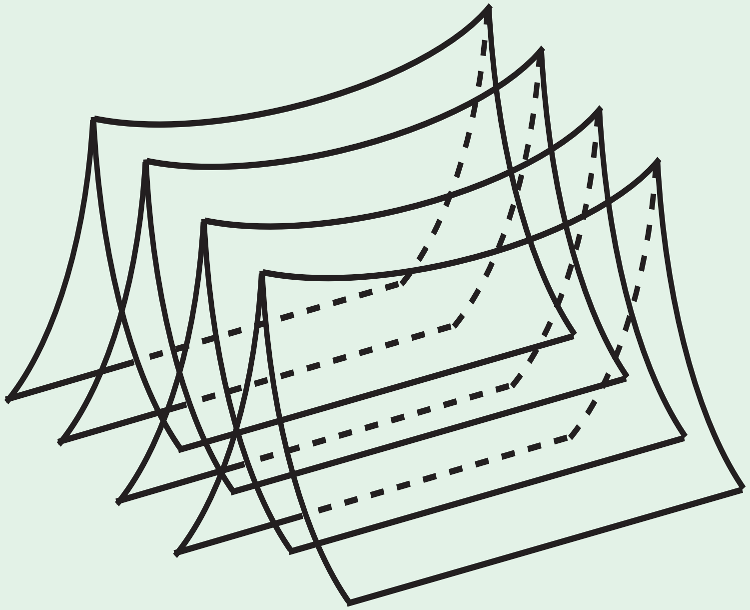
A C^∞ map-germ $\alpha : (\mathbf{R} \times \mathbf{R}, (t_0, s_0)) \to X^4$ is called a null family of null curves if there exists a C^∞ lift $\widetilde{\alpha} : (\mathbf{R} \times \mathbf{R}, (t_0, s_0)) \to Z^6$ of α such that $\alpha_*(T_{(t,s)}\mathbf{R}^2) \subset \widetilde{\alpha}(t, s)^\perp$ and $\alpha_*(T_{(t,s)}\mathbf{R} \times \{s\}) \subset \widetilde{\alpha}(t, s)$, for any $(t, s) \in \mathbf{R} \times \mathbf{R}$ nearby (t_0, s_0).



Then a null frontal hypersurface is obtained as the union of null tangent surfaces of null tangent curves $\alpha_s(t) := \alpha(t, s)$.

Example. ((1, 3) flat model.)
$\pi_X : Z^6 \to X^4, \pi_X(t, z_1, z_2, z_3, \theta_1, \theta_2) = (t, z_1, z_2, z_3)$,
$\pi_Y : Z^6 \to Y^5$,
$\pi_Y(t, z_1, z_2, z_3, \theta_1, \theta_2) = (z_1 - t\theta_1, z_2 - t\theta_2, z_3 - \frac{1}{2}t(\theta_1^2 + \theta_2^2), \theta_1, \theta_2)$.
The quadratic cone C is given by $dtdz_3 - \frac{1}{2}\{(dz_1)^2 + (dz_2)^2\} = 0$.
The contact structure on Y is given by $D : dy_3 - y_4 dy_1 - y_5 dy_2 = 0$.
Let $\beta : (\mathbf{R}^2, 0) \to Y$ be a Legendre map defined by
$\beta(u_1, u_2) = (y_1, y_2, y_3, y_4, y_5) = (u_1,\ \frac{1}{2}u_2^2,\ \frac{1}{6}u_2^3,\ 0,\ \frac{1}{2}u_2)$.
Then we have the null frontal $f = \pi_X \pi_Y^{-1}\beta : (\mathbf{R}^3, 0) \to X$,
$f(u_1, u_2, t) = (t,\ u_1,\ \frac{1}{2}u_2^2 + \frac{1}{2}u_2 t,\ \frac{1}{6}u_2^3 + \frac{1}{6}u_2^2 t)$.
The singular value set of f is parametrized by
$\alpha : (\mathbf{R} \times \mathbf{R}, (0, 0)) \to X,\ \alpha(u_1, u_2) = (-2u_2, u_1, -\frac{1}{2}u_2^2, -\frac{1}{12}u_2^3)$,
which is a null family of null curves. Then the null frontal f is obtained from α, and f is a union of tangent surfaces of null curves with parameter u_2, which is diffeomorphic to $CE \times \mathbf{R}$.

An example of singular null hypersurface generated by a null family of null curves in a Lorentz 4-manifold.



$$\subset X^4$$

ご清聴ありがとうございます.

Thank you for your attention.