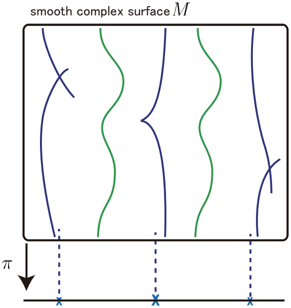
Splittings of singular fibers and vanishing cycles



Takayuki OKUDA

the University of Tokyo

Kanazawa University
Satellite Plaza

January 18, 2017

Degenerations and their splitting deformations

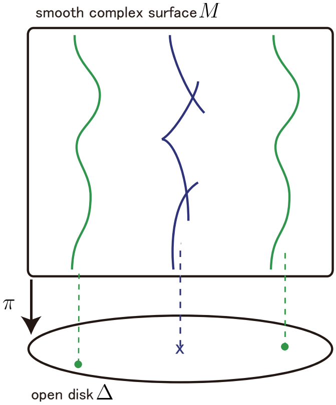
Degeneration of Riemann surfaces

M : a smooth complex surface Δ : the unit disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map s.t.

- $X_s := \pi^{-1}(s)$ $(s \neq 0)$ **are smooth curves of genus** g.
- $X_0 := \pi^{-1}(0)$ **is a singular fiber**.

($\Longleftrightarrow$ 0 is a unique critical value.)



$\pi : M \to \Delta$ is called
a **degeneration** (or, degenerating family)
of Riemann surfaces of genus g.

Regard X_0 as the divisor defined by π:

$$X_0 = \sum m_i \Theta_i,$$

where Θ_i is an irreducible component
with **multiplicity** m_i.

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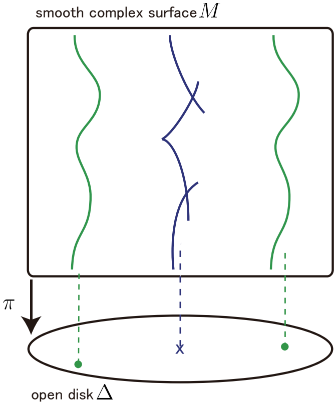

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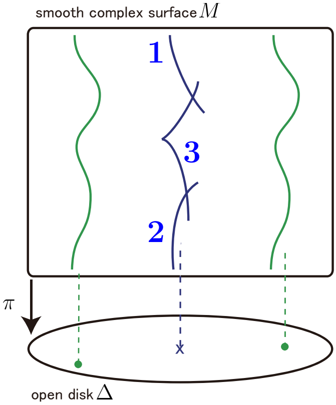

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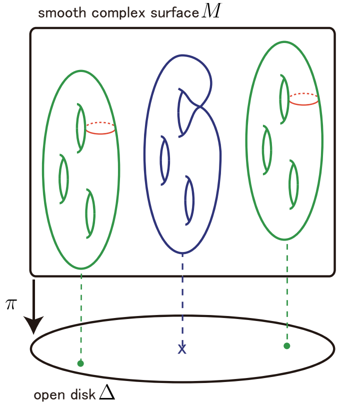

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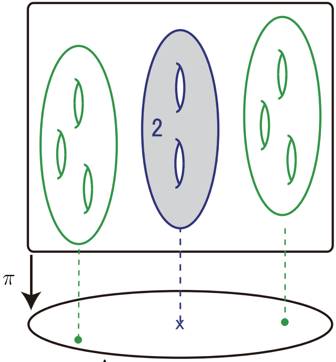

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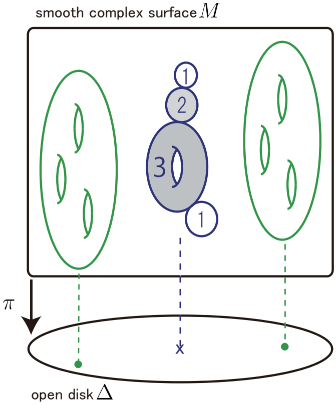

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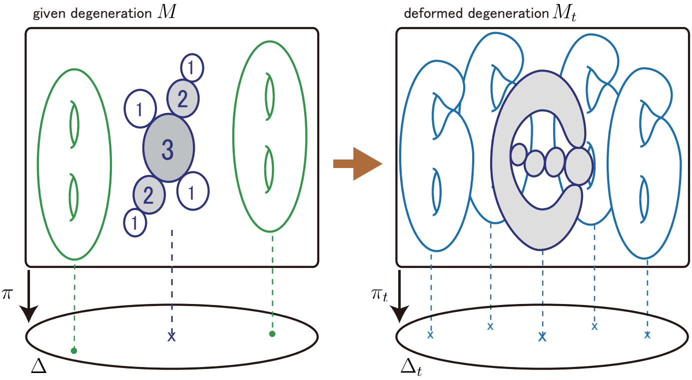
Splitting of singular fibers

$\pi : M \to \Delta$: a degeneration w/ **singular fiber X_0**

$\{\pi_t : M_t \to \Delta\}$: a family of deformations of $\pi : M \to \Delta$
i.e. $\pi_0 : M_0 \to \Delta$ coincides with $\pi : M \to \Delta$.



If π_t $(t \neq 0)$ has **k singular fibers $X_{s_1}, \ldots, X_{s_k}$**, $k \geq 2$,
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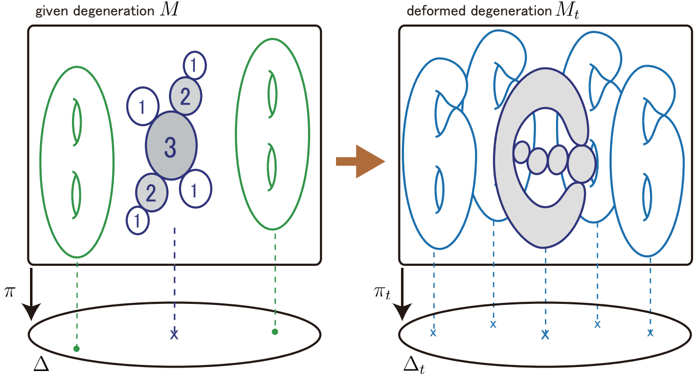

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How to construct splittings

- **Double covering method**
  - Moishezon (genus 1 case), Horikawa (genus 2 case), Arakawa-Ashikaga (hyperelliptic case)
- **Barking deformation**
  - Takamura (some criterion)

Fact (**Atoms** of singular fibers)
(1) A **Lefschetz fiber** and (2) a **multiple smooth fiber** admit no splittings (i.e. any deformation is equisingular).

Conjecture (from a topological viewpoint)
Every singular fiber can split into singular fibers each of which is (1) or (2), in finite steps of deformations.

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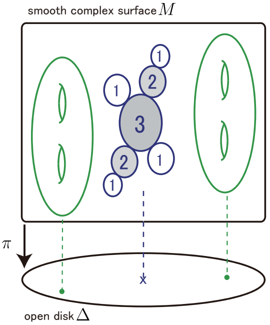
Top. equivalent degenerations are deformation equivalent.



The top. classes of degenerations are completely determined by their **topological monodromies**.

Theorem

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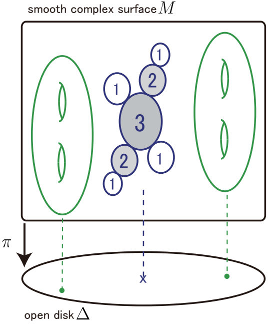

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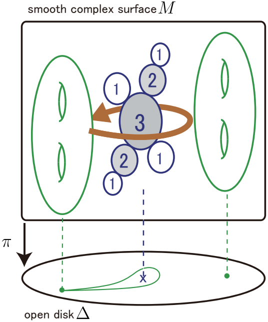

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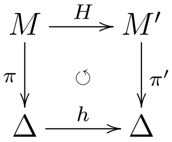

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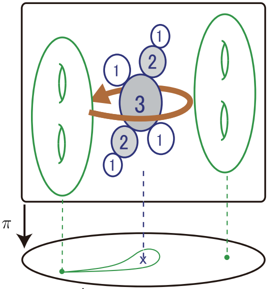

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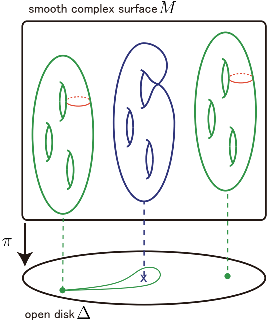
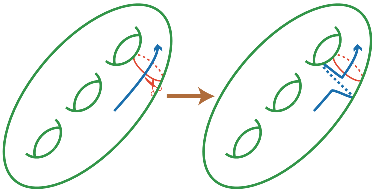
Lefschetz fiber



Right-handed Dehn twist



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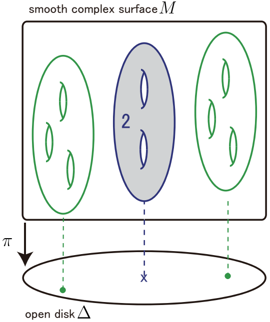
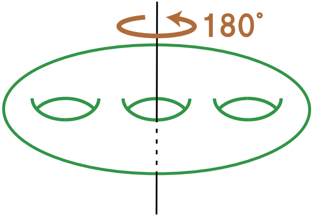


Periodic mapping class w/o multiple points



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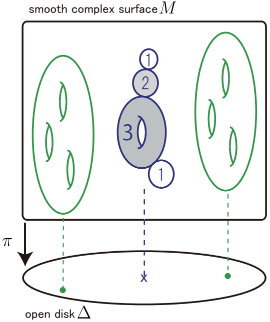
Stellar fiber



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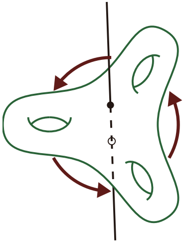

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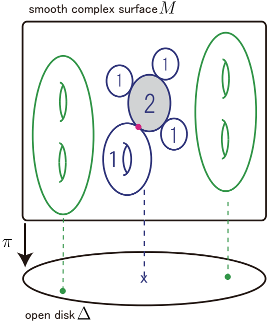
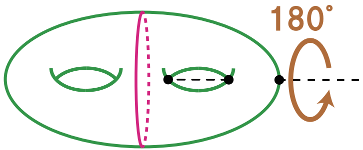
Singular fiber



Pseudo-periodic mapping class of negative twist



Splittability

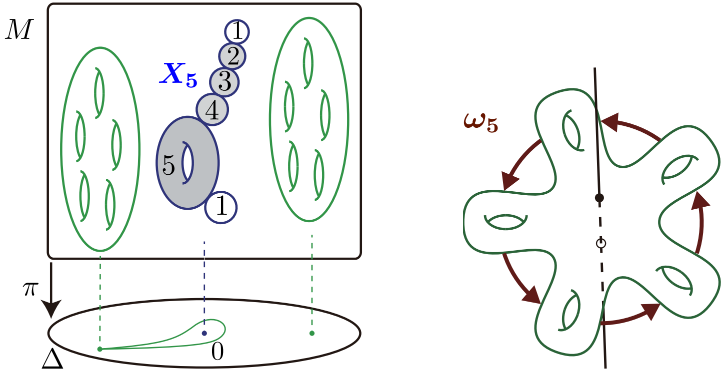
into Lefschetz fibers

Degeneration of propeller surfaces

A **propeller surface** is a Riemann surface Σ_g of genus $g \geq 2$ equipped with $\mathbb{Z}_g$-action s.t. $\Sigma_g/\mathbb{Z}_g$ has genus 1.

$\boldsymbol{\omega_g}$: a **propeller automorphism**

$\boldsymbol{X_g}$: the singular fiber with monodromy $\boldsymbol{\omega_g}$

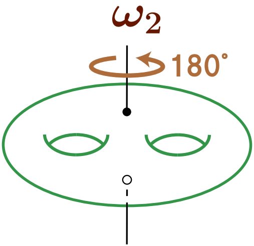


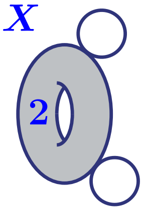
Theorem (Y. Matsumoto)

$\pi : M \to \Delta$: the degeneration of Riemann surfaces of genus 2
with monodromy ω_2

Then its singular fiber X_2 can split into **four** Lefschetz fibers.

Moreover, their vanishing cycles are as depicted below.





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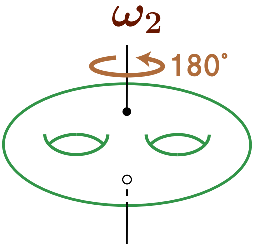
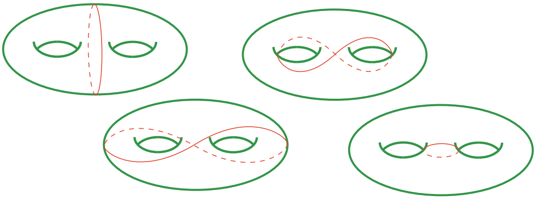
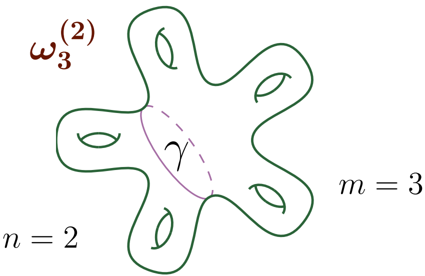
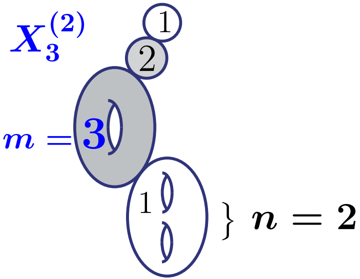








γ : a separating simple loop on Σ_g

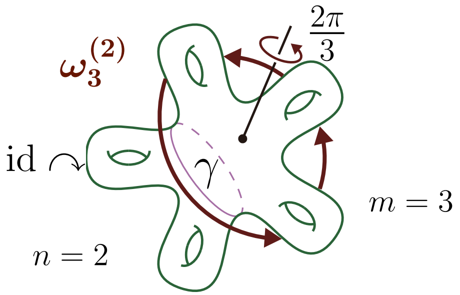
s.t. $\Sigma_g \setminus \gamma = \Sigma_{m,1} \coprod \Sigma_{n,1} \quad (g = m + n, m \geq 1, n \geq 0)$

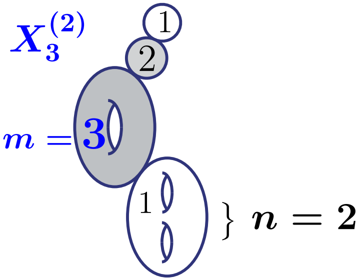
$\omega_m^{(n)}$: a psudo-periodic map satisfying

- $\omega_m^{(n)}|_{\Sigma_{m,1}} \sim$ a periodic map with a fixed pt of rot angle $\frac{2\pi}{m}$.
- $\omega_m^{(n)}|_{\Sigma_{n,1}} \sim \mathrm{id}.$

NOTE: $(\omega_m^{(n)})^m = \tau_\gamma$

$X_m^{(n)}$: the singular fiber with monodromy $\omega_m^{(n)}$





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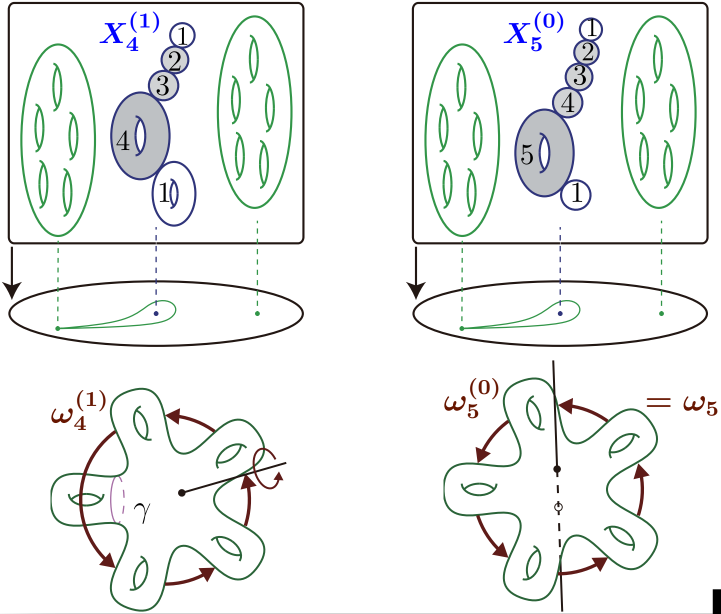

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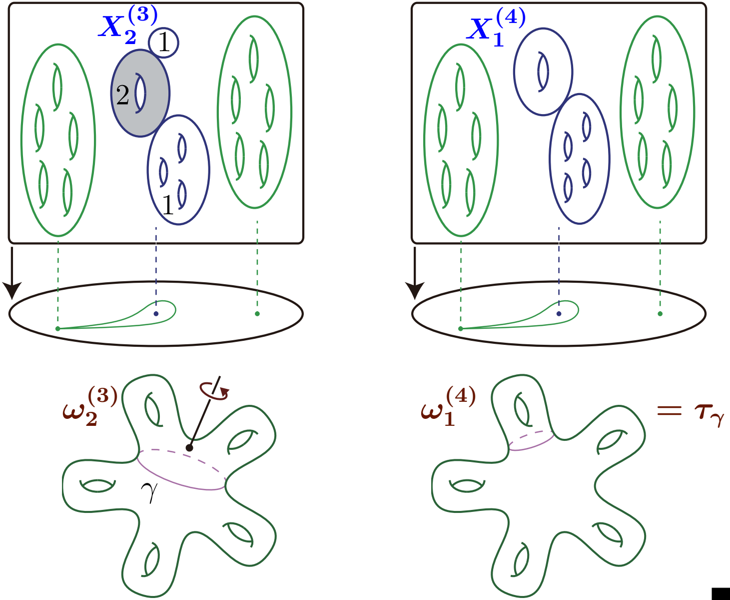
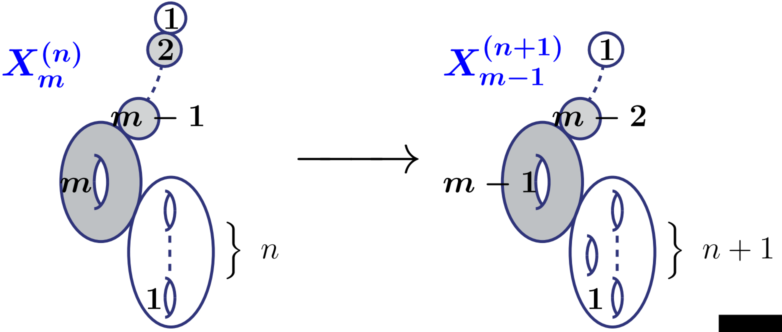


Theorem (O-Takamura)

1. For any $m \geq 2$, $n \geq 0$, the singular fiber $X_m^{(n)}$ can split into $X_{m-1}^{(n+1)}$ and three Lefschetz fibers.
2. For any $g \geq 2$, we have the following sequence:
   $$X_g^{(0)} \longrightarrow X_{g-1}^{(1)} \longrightarrow \cdots \longrightarrow X_2^{(g-2)} \longrightarrow X_1^{(g-1)},$$
   where "$A \to B$" means "A splits into B and 3 Lefschetz fibers".



Theorem (O-Takamura)

1. For any $m \geq 2$, $n \geq 0$, the singular fiber $X_m^{(n)}$ can split into $X_{m-1}^{(n+1)}$ and three Lefschetz fibers.
2. For any $g \geq 2$, we have the following sequence:
$$X_g^{(0)} \longrightarrow X_{g-1}^{(1)} \longrightarrow \cdots \longrightarrow X_2^{(g-2)} \longrightarrow X_1^{(g-1)},$$
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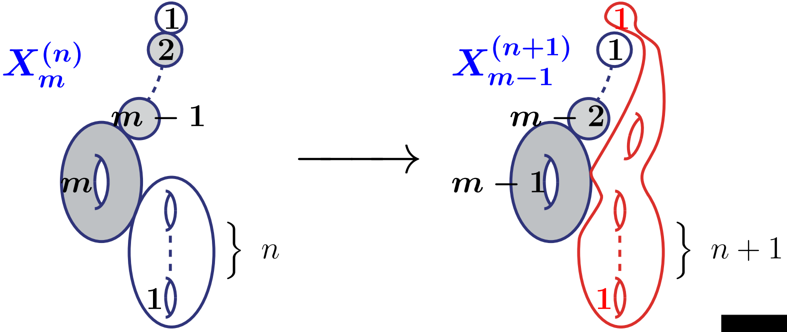

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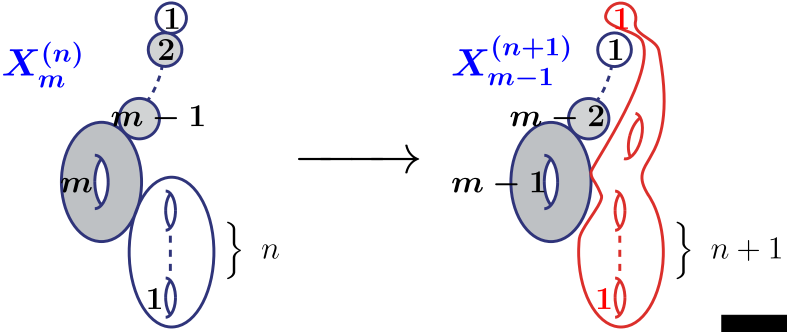
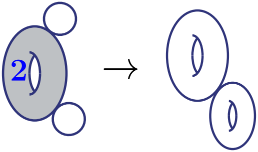


$$X_m^{(n)} \longrightarrow X_{m-1}^{(n+1)}$$

$g = 2$



$g = 3$

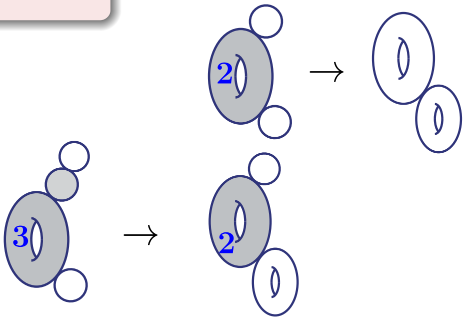
$g = 4$

$$\boldsymbol{X_m^{(n)}} \longrightarrow \boldsymbol{X_{m-1}^{(n+1)}}$$

$g = 2$

$g = 3$

$g = 4$



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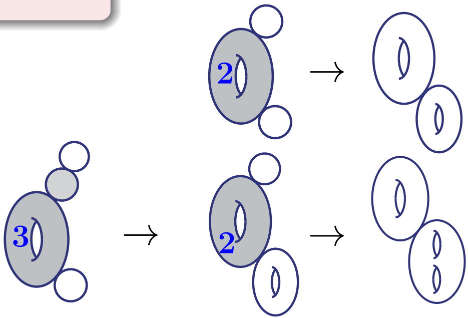

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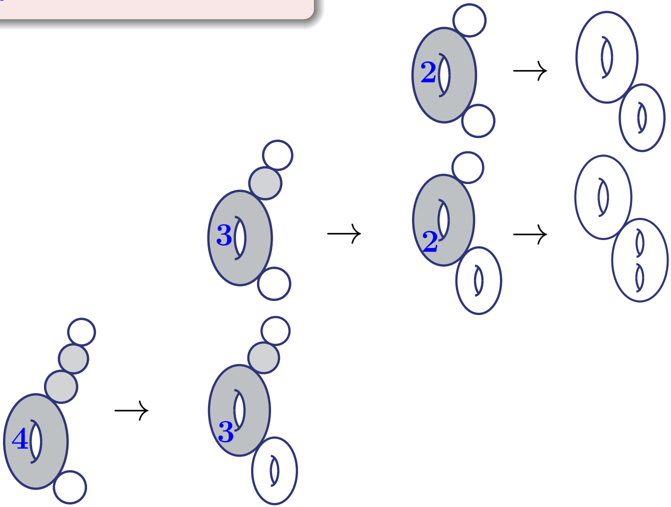

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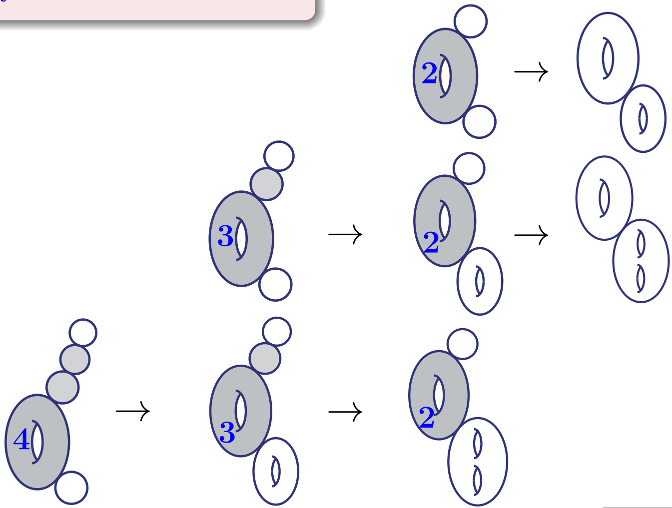

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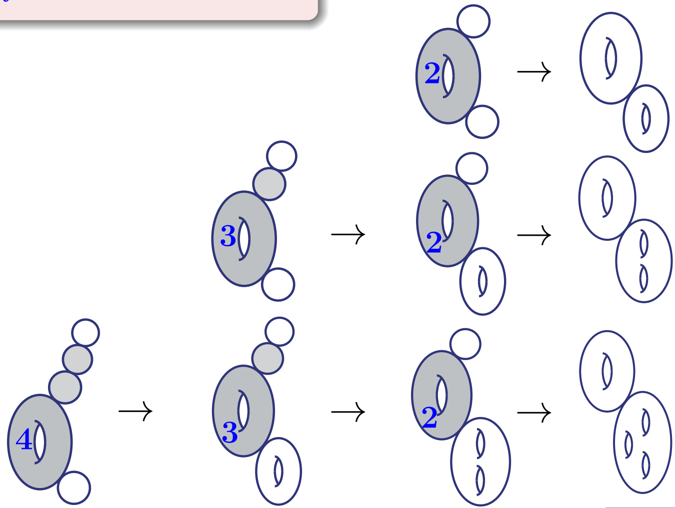
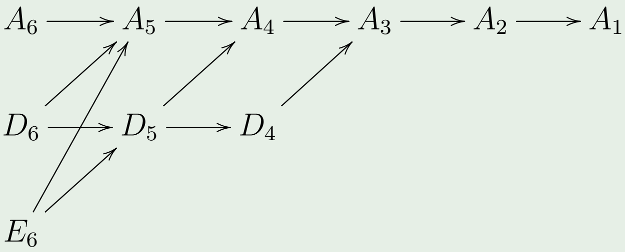


Remark of Theorem

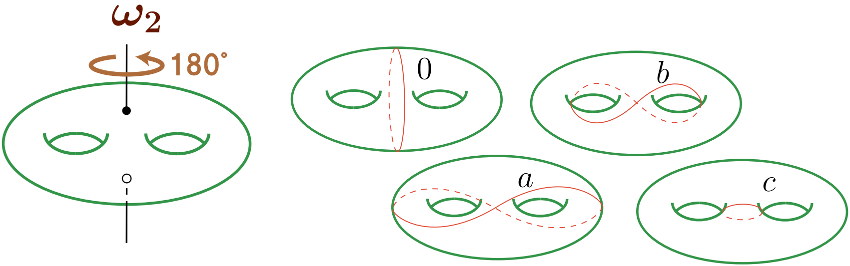
1. Generalization of Matsumoto's splitting for genus 2.
2. Analogous to adjacency diagrams of singularities:



3. A splitting of a singular fiber into Lefschetz fibers gives a **Dehn-twist expression** of its topological monodromy.

Theorem (Y. Matsumoto)

1. The singular fiber X_2 can split into **four** Lefschetz fibers, and their vanishing cycles are as depicted below.
2. $\omega_2 = \tau_0 \circ \tau_a \circ \tau_b \circ \tau_c$.



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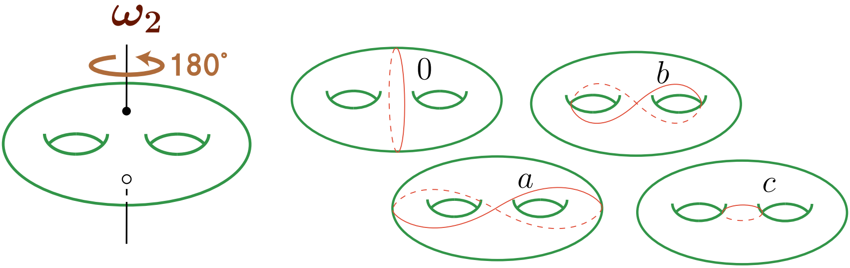
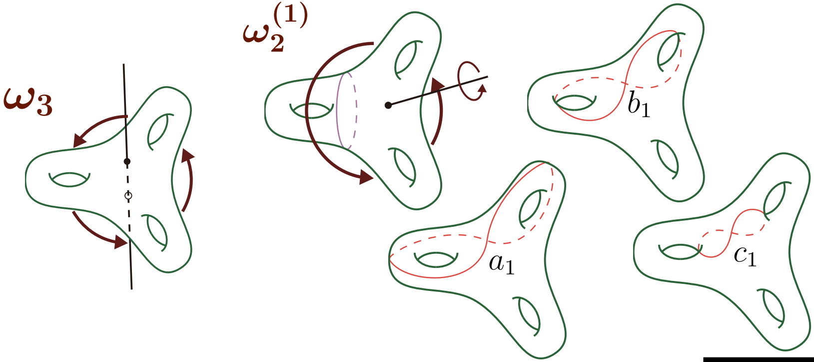


Proposition

1. $\boldsymbol{X_3^{(0)}}$ can split into $\boldsymbol{X_2^{(1)}}$ and three Lefshetz fibers, and their vanishing cycles are as depicted below.
2. $\omega_3 = \omega_2^{(1)} \circ \tau_{a_1} \circ \tau_{b_1} \circ \tau_{c_1}$.



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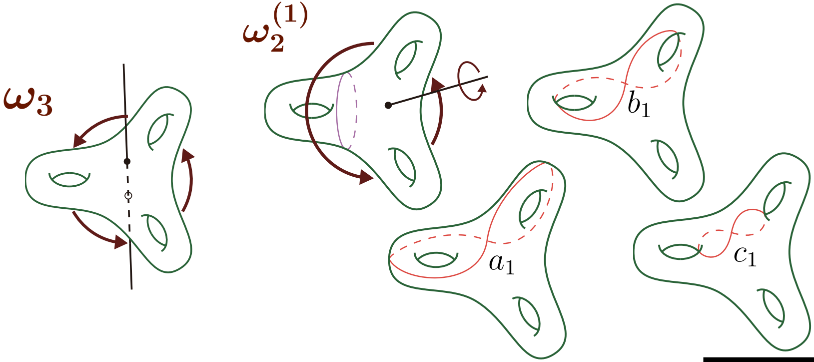

Proposition

1. $\boldsymbol{X_2^{(1)}}$ can split into four Lefshetz fibers (including $\boldsymbol{X_1^{(2)}}$), and their vanishing cycles are as depicted below.
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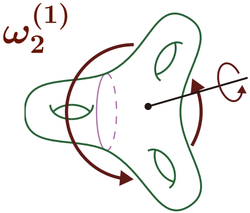

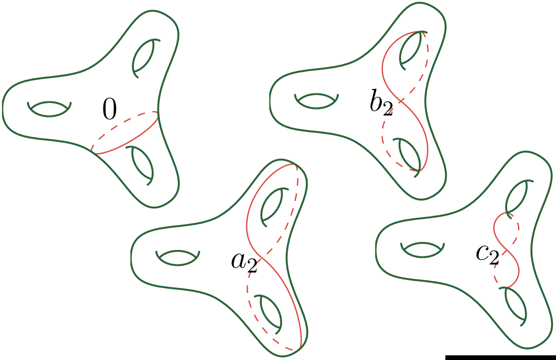

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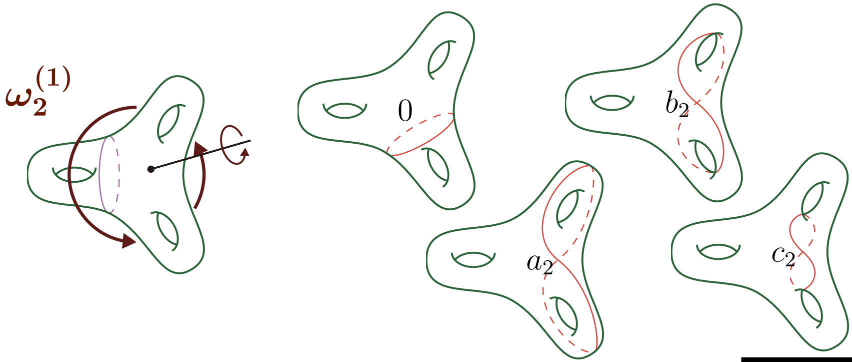
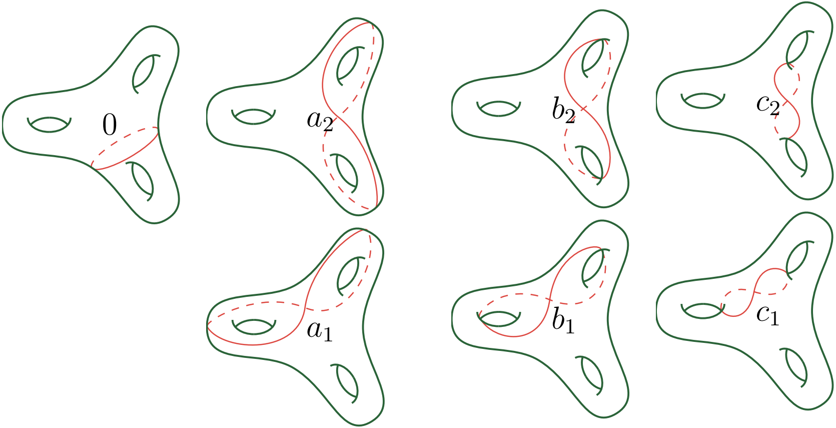


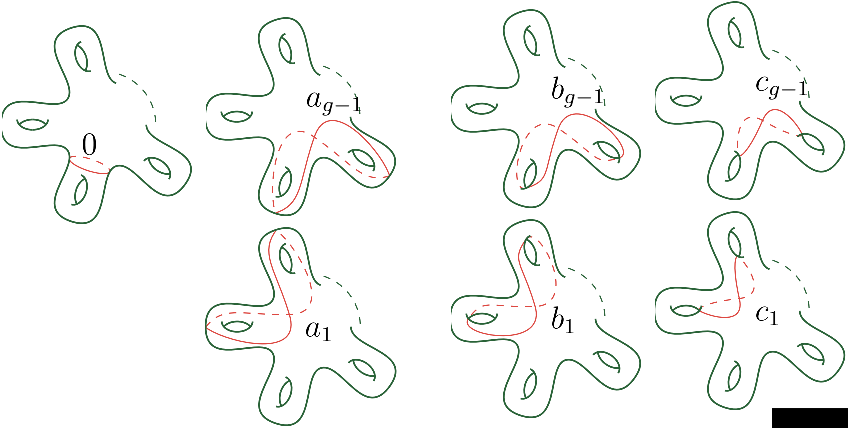
Proposition

$\omega_3 = \tau_0 \circ (\tau_{a_2} \circ \tau_{b_2} \circ \tau_{c_2}) \circ (\tau_{a_1} \circ \tau_{b_1} \circ \tau_{c_1})$.



Theorem

$$\omega_g = \tau_0 \circ (\tau_{a_{g-1}} \circ \tau_{b_{g-1}} \circ \tau_{c_{g-1}}) \circ \cdots \circ (\tau_{a_1} \circ \tau_{b_1} \circ \tau_{c_1}).$$



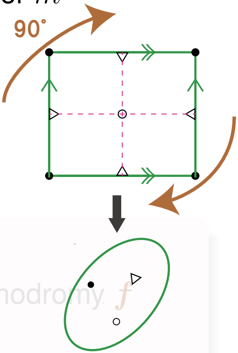
$\boldsymbol{f} \in \mathrm{MCG}(\Sigma_g)$: a periodic mapping class of order m

$\boldsymbol{b(f)} := \#\{$branch points of $\Sigma_g \to \Sigma_g/f\}$

$\boldsymbol{p(f)} := \#\{$**propeller points** of $f\} \leq h(f)$

> fixed points of f with rotation angle $\pm 1/m$

$\boldsymbol{r(f)} := \sum q_j/\ell_j \in \mathbb{Z}_+$: the total valency sum



Theorem (O)

X : the singular fiber equipped with periodic monodromy f.
Suppose f satisfies at least one of the following:

- $b(f) - p(f) \leq r(f)$.
- $b(f) - p(f) \leq 2$, $r(f) = 1$ and $\mathrm{genus}(\Sigma_g/f) = 0$.

Then X can splits into Lefschetz fibers.

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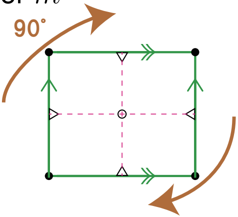

Theorem (O)

$\boldsymbol{X}$: the singular fiber equipped with periodic monodromy $\boldsymbol{f}$

Suppose $\boldsymbol{f}$ satisfies at least one of the following:

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Then $\boldsymbol{X}$ can splits into Lefschetz fibers.

Remark of Theorem

1. If $m = 2, 3$, then $\boldsymbol{X}$ can split into Lefschetz fibers.
2.

genus	1	2	3	4	5	6
# of periodic m.c.	**8**	**17**	**47**	**72**	**76**	**203**
# of periodic m.c. satisfying $(*)$	**8**	**14**	**30**	**41**	**35**	
# of powers of periodic m.c. satisfying $(*)$	**8**	**16**	**45**	**66**	**66**	

Thank you for your attention.