微分方程式の幾何学と分類問題 II

by

Kazuhiro Shibuya
(Hiroshima University)

- Differential Equation
- Differential geometry (Differential systems)
- Lie algebra

R : smooth manifold

$D \subset TR$: subbundle

(R, D) : **differential system**, distribution or Pfaffian system

rank D := the rank of D as a vector bundle

- (local) isomorphism ϕ :

  diffeomorphism $\phi : (R_1, D_1) \to (R_2, D_2) \quad , \quad \phi_*(D_1) = D_2$

Example (Contact manifold: (J, C))

$J := \mathbb{R}^{2n+1} : (x_1, \ldots, x_n, y, p_1, \ldots p_n)$
Put $\theta := dy - \Sigma_i p_i dx$
$C := \{\theta = 0\} = \{X \in TJ \mid \theta(X) = 0\},$

Example (The canonical contact system on k-jet space of n independent and m dependent variables: $(J^k(\mathbb{R}^n, \mathbb{R}^m), C^k)$)

$J^k(\mathbb{R}^n, \mathbb{R}^m) : (x_i, y^\alpha, p_I^\alpha) \;\; (1 \leq |I| \leq k)$,
$C^k = \{\varpi_I^\alpha = 0 \quad (0 \leq |I| \leq k-1,\ 1 \leq \alpha \leq m)\}$,

where I is a multi-index,

$$\varpi_0^\alpha = dy^\alpha - \textstyle\sum_{i=1}^n p_i^\alpha dx_i \ , \ \varpi_I^\alpha = dp_I^\alpha - \textstyle\sum_{i=1}^n p_{Ii}^\alpha dx_i .$$

- **Derived System**

$$\partial D := D + [D, D]$$

$$(\partial D \ : \ \partial\mathcal{D} := \mathcal{D} + [\mathcal{D}, \mathcal{D}] \text{ where } \mathcal{D} = \Gamma(D))$$

- The derived system of D is not always a subbundle of TR.
- $D = \partial D \iff D$ is completely integrable.

i**-th Derived System**: $\partial^i D := \partial(\partial^{i-1} D)$

- Assume $\partial^i D$ are subbundles $(\forall i)$
- $\exists i_0$ s.t.

$$D \subset \cdots \subset \partial^{i_0-1} D \subset \partial^{i_0} D = \partial^{i_0+1} D = \cdots \subset TR$$

then, $\partial^{i_0} D$ is the smallest completely integrable subbundle which contains D

- **Cauchy characteristic system** $Ch(D)$ of D

$$Ch(D)(x) := \{X(x) \in D(x) \mid X \in \mathcal{D}\ ,\ [X,Y] \in \mathcal{D}\ (\forall Y \in \mathcal{D})\}$$

where $\mathcal{D} := \Gamma(D)$.

$Ch(D)$:constant rank $\Rightarrow$ $Ch(D)$ is completely integrable

Remark

The Cauchy characteristic system is the biggest completely integrable subbundle contained in D

Motivation

Geometry of 2nd order 2 independent and 1 dependent variables single PDE was well studied by Lie, Darboux, Goursat, Monge, Cartan, Tresse, etc. around 1900.

Example

$r - t = 0$ (wave equation, hyperbolic)
$r - q = 0$ (heat equation, parabolic)
$r + t = 0$ (Laplace's equation, elliptic)

where, x, y, u, p, q, r, s, t are the classical notations

After that, the theory is developed by
Bryant, Chern, Gardner, Goldschmidt, and Griffiths, etc. (“MSRI group”)
and Tanaka, Morimoto and Yamaguchi, etc. (“Tanaka school”)

Today’s goal :

- to give a classification for Darboux integrable f-Gordon equation (joint work with Yoshimoto)

The Cartan-Kahler theorem states about the existence of analytic solutions for PDE and a generalization of the Frobenius theorem and the Cauchy-Kowalewski theorem.

- If a PDE have **torsion**, then there exist no solutions for the PDE
- If a PDE is **involutive**, then there exists a solution for the PDE
- If a PDE is **not involutive**, then we do not know anything for solutions for the PDE

Remark

All PDEs are divided into the 3 cases.

2nd order PDE

$J^2(\mathbb{R}^2, \mathbb{R}^1) \cong \mathbb{R}^8$: 2-jet space

(x, y, u, p, q, r, s, t) : canonical coordinate system.

$$C^2 = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} \subset TJ^2(\mathbb{R}^2, \mathbb{R}^1)$$

where

$$\begin{aligned} \varpi_0 &= du - pdx - qdy \\ \varpi_1 &= dp - rdx - sdy \\ \varpi_2 &= dq - sdx - tdy. \end{aligned}$$

Then,

$$R := \{F(x, y, u, p, q, r, s, t) = 0\} \subset J^2(\mathbb{R}^2, \mathbb{R})$$

$$D := C^2|_R = \{\iota^*\varpi_0 = \iota^*\varpi_1 = \iota^*\varpi_2 = 0\} \subset TR$$

where, $\iota : R \to J^2(\mathbb{R}^2, \mathbb{R})$ is the inclusion

$\dim R = 7$, $\operatorname{rank} D = 4$

- (R, D) is called
  the **associated differential system** to the PDE : $F = 0$.
- All 2nd order PDE with 2 indep. and 1 dep. variables are involutive.
- The degree of freedom of the solution of the PDE is 2 functions of 1 variable.

Definition For $w \in R = \{F = 0\}$,

$$\Delta(w) := F_r F_t - \frac{1}{4} F_s^2(w) < 0 \iff w \text{ is hyperbolic point}$$
$$\Delta(w) := F_r F_t - \frac{1}{4} F_s^2(w) = 0 \iff w \text{ is parabolic point}$$
$$\Delta(w) := F_r F_t - \frac{1}{4} F_s^2(w) > 0 \iff w \text{ is elliptic point}$$

In this talk, we assume the types do not depend on any point w

To be Hyp., Par. or Eii. is invariant under contact isomorphisms.

Example (hyperbolic)

$$\frac{\partial^2 u}{\partial x \partial y} = 0$$

$$R := \{s = 0\} \subset J^2(\mathbb{R}^2, \mathbb{R}) \qquad D = C^2|_\Sigma$$

$$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$$

$$\begin{aligned} d\varpi_0 &\equiv dx \wedge \varpi_1 + dy \wedge \varpi_2 \qquad \mod \ \varpi_0, \\ d\varpi_1 &\equiv dx \wedge dr \qquad \mod \ \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv dy \wedge dt \qquad \mod \ \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

where

$\{\varpi_0, \varpi_1, \varpi_2, dx, dy, dr, dt\}$:coframe on Σ.

Example (parabolic)

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$

$$\begin{aligned} d\varpi_0 &\equiv dx \wedge \varpi_1 + dy \wedge \varpi_2 \qquad \mod \ \varpi_0, \\ d\varpi_1 &\equiv \qquad\qquad dy \wedge ds \qquad \mod \ \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv dx \wedge ds + dy \wedge dt \qquad \mod \ \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

where

$\{\varpi_0, \varpi_1, \varpi_2, dx, dy, ds, dt\}$:coframe on Σ.

Example (elliptic)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$

$$\begin{aligned} d\varpi_0 &\equiv dx \wedge \varpi_1 + dy \wedge \varpi_2 && \mod \ \varpi_0, \\ d\varpi_1 &\equiv dx \wedge dr + dy \wedge ds && \mod \ \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv dx \wedge ds - dy \wedge dr && \mod \ \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

where

$\{\varpi_0, \varpi_1, \varpi_2, dx, dy, dr, ds\}$:coframe on Σ.

Fact

For regular single PDE (R, D),

hyperbolic case

There exists a local coframe $\{\theta_0, \theta_1, \theta_2, \eta_1, \eta_2, \pi_1, \pi_2\}$ around w such that followings hold at w $(w \in R)$.

$$D = \{\theta_0 = \theta_1 = \theta_2 = 0\}$$

$$\begin{aligned} d\theta_0 &\equiv 0 && \mod \theta_0, \theta_1, \theta_2, \\ d\theta_1 &\equiv \eta_1 \wedge \pi_1 && \mod \theta_0, \theta_1, \theta_2, \\ d\theta_2 &\equiv \eta_2 \wedge \pi_2 && \mod \theta_0, \theta_1, \theta_2. \end{aligned}$$

parabolic case

There exists a local coframe $\{\theta_0, \theta_1, \theta_2, \eta_1, \eta_2, \pi_1, \pi_2\}$ around w such that followings hold at w.

$$\begin{aligned}
d\theta_0 &\equiv 0 && \mod \theta_0, \theta_1, \theta_2, \\
d\theta_1 &\equiv \eta_1 \wedge \pi_1 && \mod \theta_0, \theta_1, \theta_2, \\
d\theta_2 &\equiv \eta_1 \wedge \pi_2 + \eta_2 \wedge \pi_1 && \mod \theta_0, \theta_1, \theta_2.
\end{aligned}$$

elliptic case

There exists a local coframe $\{\theta_0, \theta_1, \theta_2, \eta_1, \eta_2, \pi_1, \pi_2\}$ around w such that followings hold at w.

$$\begin{aligned}
d\theta_0 &\equiv 0 && \mod \theta_0, \theta_1, \theta_2, \\
d\theta_1 &\equiv \eta_1 \wedge \pi_1 + \eta_2 \wedge \pi_2 && \mod \theta_0, \theta_1, \theta_2, \\
d\theta_2 &\equiv \eta_1 \wedge \pi_2 - \eta_2 \wedge \pi_1 && \mod \theta_0, \theta_1, \theta_2.
\end{aligned}$$

f-Gordon equations

$$\frac{\partial z}{\partial x \partial y} = f(z) \quad (f : \text{function})$$

- Wave equation $z_{xy} = 0$.
- Liouville equation $z_{xy} = e^z$.
- Klein-Gordon equation $z_{xy} = z$.
- sine-Gordon equation $z_{xy} = \sin z$.

Remark

- f-Gordon equations are hyperbolic

- The set of all f-Gordon equations are not closed under the action of contact isomorphisms.

$$\begin{aligned} G &:= \{\text{contact iso}\}, \\ H &:= \{\text{hyperbolic PDE}\}, \\ X &:= \{f\text{–Gordon equation}\}. \end{aligned}$$

$G \curvearrowright H$ is closed

For $X \subset H$, $G \curvearrowright X$ is not closed.

Theorem(S-Yoshimoto)
For f-Gordon equation $(R = \{s - f(z) = 0\}, D)$,

(R, D) is Darboux integrable $\iff f(z) = C$ or $f = C_2 \mathrm{e}^{C_1 z}$ $(C, C_1 \neq 0, C_2 \neq 0)$.

Corollary(S-Yoshimoto)
For a PDE (R, D) which is contact equivalent to f-Gordon equation ,

(R, D) is Darboux integrable $\iff$ the PDE is contact equivalent to $f(z) = 0$ or $f = \mathrm{e}^z$

Corollary(S-Yoshimoto)
Monge characteristic systems $M_i \, (i = 1, 2)$ are regular.

Monge Method

$z_{xy} = 0.$

$$\begin{aligned} R &:= \{F := s = 0\}, \\ D &:= C|_R := \{\iota^*\varpi_0 = \iota^*\varpi_1 = \iota^*\varpi_2 = 0\} \ (\iota : R \hookrightarrow J^2(2,1)). \end{aligned}$$

$\theta_0 := \iota^*\varpi_0,$

$$\begin{aligned} \theta_0 &= \iota^*(dz - pdx - qdy) \\ &= \iota^*dz - (\iota^*p)(\iota^*dx) - (\iota^*q)(\iota^*dy) \\ &= d(\iota^*z) - (\iota^*p)d(\iota^*x) - (\iota^*q)d(\iota^*y) \\ &= dz - pdx - qdy. \\ \therefore dz &= \theta_0 + pdx + qdy. \end{aligned}$$

$\theta_1 := \iota^*\varpi_1, \ \ \theta_2 := \iota^*\varpi_2,$

$$\begin{aligned} \theta_1 &= dp - rdx, \ dp = \theta_1 + rdx. \\ \theta_2 &= dq - tdy, \ dq = \theta_2 + tdy. \end{aligned}$$

Structure equation

$$
\begin{aligned}
d\theta_1 &= -dr \wedge dx \\
&= \pi_1 \wedge \omega_1 \quad (\pi_1 := -dr,\ \omega_1 := dx). \\
d\theta_2 &= -dt \wedge dy \\
&= \pi_2 \wedge \omega_2 \quad (\pi_2 := -dt,\ \omega_2 := dy). \\
d\theta_0 &= -dp \wedge dx - dq \wedge dy \\
&= -(\theta_1 + r dx) \wedge dx - (\theta_2 + t dy) \wedge dy \\
&= -\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2.
\end{aligned}
$$

where, coframe$\{\theta_0, \theta_1, \theta_2, \pi_1, \pi_2, \omega_1, \omega_2\}$.

$$M_i := \{\theta_0 = \theta_1 = \theta_2 = \pi_i = \omega_i = 0\} \ (i = 1, 2)$$

are called **Monge characteristic systems**.

(R, D) is **Darboux integrable** $:\Longleftrightarrow$

$$^{\exists}F_i \subset TR \ \text{ s.t. } \begin{pmatrix} F_i \text{ is completely integrable} \\ \operatorname{rank} {F_i}^{\perp} = 2 \mod D^{\perp} \\ M_i \subset F_i \end{pmatrix} \quad (i = 1, 2) \qquad (\#)$$

Remark

- Monge characteristic systems are invariant subsystems.
- Since $M_i \subset F_i$, $\partial M_i \subset \partial F_i = F_i$.

Wave equation (R, D) is Darboux integrable

For M_1,

$$\begin{aligned} \partial M_1 &= \{\theta_0 = \theta_1 = \pi_1 = \omega_1 = 0\}, \\ \partial^2 M_1 &= \{\theta_1 = \pi_1 = \omega_1 = 0\} = \partial^3 M_1. \end{aligned}$$

For M_2,

$$\partial^2 M_2 = \{\theta_2 = \pi_2 = \omega_2 = 0\} = \partial^3 M_2.$$

Hence, if we put

$$F_i = \{\pi_i = \omega_i = 0\} \ (i = 1, 2)$$

then, (#) is satisfied.

$$F_1 = \{dr = dx = 0\},\ F_2 = \{dt = dy = 0\}$$

Take

$$r = f(x), \quad t = g(y) \quad (f, g : \text{arbitrary functions})$$

$$R_{f,g} := \{r = f(x),\ t = g(y)\}, \quad D_{f,g} := D|_{R_{f,g}}.$$

From $D_{f,g} = \partial D_{f,g}$, $D_{f,g}$ is completely integrable. We have

$$D_{f,g} = \{dh_0 = dh_1 = dh_2 = 0\}$$

$$\begin{aligned} h_0 &= z - px - qy + (x-1)\bar{f} + (y-1)\bar{g}, \\ h_1 &= p - \bar{f}, \\ h_2 &= q - \bar{g} \end{aligned}$$

where, $\bar{f}(x) := \int f(x)dx,\ \bar{g}(y) := \int g(y)dy$.

$$\begin{cases} z - px - qy + (x-1)\bar{f} + (y-1)\bar{g} = 0 \\ p - \bar{f} = 0 \\ q - \bar{g} = 0 \end{cases}$$

$$z = \bar{f}(x) + \bar{g}(y) \quad (\bar{f}, \bar{g} : \text{arbitrary functions}).$$

$$R := \{F := s - f(z) = 0\}, \quad D := C|_R.$$

$$\begin{aligned}
\theta_0 &:= \iota^*\varpi_0 = dz - pdx - qdy, \quad dz = \theta_0 + pdx + qdy, \\
\theta_1 &:= \iota^*\varpi_1 = dp - rdx - fdy, \quad dp = \theta_1 + rdx + fdy, \\
\theta_2 &:= \iota^*\varpi_2 = dq - fpdx - tdy, \quad dq = \theta_2 + fdx + tdy.
\end{aligned}$$

$$\begin{aligned}
d\theta_1 &= -dr \wedge dx - df \wedge dy \\
&= -dr \wedge dx - f'dz \wedge dy \\
&= -dr \wedge dx - f'(\theta_0 + pdx + qdy) \wedge dy \\
&\equiv (pf'dy - dr) \wedge dx \mod \theta_0 \\
&\equiv \pi_1 \wedge \omega_1 \quad (\pi_1 := pf'dy - dr,\ \omega_1 := dx)
\end{aligned}$$

$$
\begin{aligned}
d\theta_2 &= -dt \wedge \omega_2 - f'(\theta_0 + q\omega_2) \wedge \omega_1 \\
&\equiv \pi_2 \wedge \omega_2 \mod \theta_0 \\
(\pi_2 &:= qf'dx - dt,\ \omega_2 := dy). \\
d\theta_0 &= -\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2.
\end{aligned}
$$

$$
\begin{aligned}
d\pi_1 &= d(pf'dy - dr) = d(pf') \wedge dy \\
&= (f'dp + pdf') \wedge dy \\
&= \{f'(\theta_1 + rdx) + pf''(\theta_0 + pdx)\} \wedge dy \\
&= \{pf''\theta_0 + f'\theta_1 + (p^2 f'' + f'r)\omega_1\} \wedge \omega_2.
\end{aligned}
$$

$$
d\pi_2 = \{pf''\theta_0 + f'\theta_2 + (q^2 f'' + f't)\omega_2\} \wedge \omega_1.
$$

Lemma

$$\begin{aligned}
d\theta_0 &= -\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2, \\
d\theta_1 &= -dr \wedge \omega_1 - f'(\theta_0 + p\omega_1) \wedge \omega_2 \\
&\equiv \pi_1 \wedge \omega_1 \mod \theta_0, \\
d\theta_2 &= -dt \wedge \omega_2 - f'(\theta_0 + q\omega_2) \wedge \omega_1 \\
&\equiv \pi_2 \wedge \omega_2 \mod \theta_0, \\
d\pi_1 &= \{pf''\theta_0 + f'\theta_1 + (p^2 f'' + f'r)\omega_1\} \wedge \omega_2, \\
d\pi_2 &= \{pf''\theta_0 + f'\theta_2 + (q^2 f'' + f't)\omega_2\} \wedge \omega_1.
\end{aligned}$$

$$M_1 := \{\theta_0 = \theta_1 = \theta_2 = \pi_1 = \omega_1 = 0\}.$$

$$\begin{aligned}
\partial M_1 &= \{\theta_0 = \theta_1 = \pi_1 = \omega_1 = 0\}. \\
\because d\theta_0 &\equiv 0 \mod M_1, \\
d\theta_1 &\equiv 0 \mod M_1, \\
d\theta_2 &\equiv \pi_2 \wedge \omega_2 \mod M_1, \\
d\pi_1 &\equiv 0 \mod M_1, \\
d\omega_1 &= 0.
\end{aligned}$$

$$\begin{aligned}
\partial^2 M_1 &= \{\theta_1 = \pi_1 = \omega_1 = 0\}. \\
\because d\theta_0 &\equiv -\theta_2 \wedge \omega_2 \mod \partial M_1, \\
d\theta_1 &\equiv 0 \mod \partial M_1, \\
d\pi_1 &\equiv 0 \mod \partial M_1, \\
d\omega_1 &= 0.
\end{aligned}$$

Calculation for $\partial^3 M_1$

From Lemma ,

$$\begin{aligned} d\theta_1 &\equiv -f'\theta_0 \wedge \omega_2 \mod \partial^2 M_1, \\ d\pi_1 &\equiv pf''\theta_0 \wedge \omega_2 \mod \partial^2 M_1. \end{aligned}$$

(i) $f' \equiv 0$, i.e., $f \equiv C\,(C:\text{costant})$,

$$\partial^3 M_1 = \{\theta_1 = \pi_1 = \omega_1 = 0\} = \partial^2 M_1.$$

Therefore, $F_1 := \{\pi_1 = \omega_1 = 0\}$ satisfies the condition (#).

(ii) $f' \neq 0$,

$\bar{\pi}_1 := pf''\theta_1 + f'\pi_1$,

$$\begin{aligned} d\bar{\pi}_1 &= d(pf'') \wedge \theta_1 + pf'' d\theta_1 + df' \wedge \pi_1 + f' d\pi_1 \\ &\equiv 0 \mod \partial^2 M_1. \end{aligned}$$

$$\partial^2 M_1 = \{\theta_1 = \bar{\pi}_1 = \omega_1 = 0\},$$

$$\partial^3 M_1 = \{\bar{\pi}_1 = \omega_1 = 0\}.$$

$\bar{\bar{\pi}}_1 := \frac{1}{f'}\bar{\pi}_1,$

$$\begin{aligned} d\bar{\bar{\pi}}_1 &= d\left(\frac{pf''}{f'}\theta_1 + \pi_1\right) = d\left(\frac{pf''}{f'}\right) \wedge \theta_1 + \frac{pf''}{f'} d\theta_1 + d\pi_1 \\ &\equiv d\left(\frac{pf''}{f'}\right) \wedge \theta_1 + \frac{pf''}{f'}(-f')\theta_0 \wedge \omega_2 + (pf''\theta_0 + f'\theta_1) \wedge \omega_2 \\ &\qquad \mod \omega_1 \\ &\equiv \left\{ d\left(\frac{pf''}{f'}\right) - f'\omega_2 \right\} \wedge \theta_1. \end{aligned}$$

$$
\begin{aligned}
d\left(\frac{pf''}{f'}\right) &= \frac{f''}{f'}dp + pd\left(\frac{f''}{f'}\right) \\
&= \frac{f''}{f'}(\theta_1 + r\omega_1 + f\omega_2) + p\left(\frac{f''}{f'}\right)'(\theta_0 + p\omega_1 + q\omega_2).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d\bar{\bar{\bar{\pi}}}_1 &\equiv \left\{\frac{f''}{f'}f\omega_2 + p\left(\frac{f''}{f'}\right)'(\theta_0 + q\omega_2) - f'\omega_2\right\} \wedge \theta_1 \quad \mod \quad \omega_1 \\
&\equiv \left\{p\left(\frac{f''}{f'}\right)'\theta_0 + \left(\frac{f''f}{f'} + pq\left(\frac{f''}{f'}\right)' - f'\right)\omega_2\right\} \wedge \theta_1.
\end{aligned}
$$

$$\begin{aligned} d\bar{\bar{\pi}}_1 \equiv 0 \mod \omega_1 &\Longleftrightarrow p\left(\frac{f''}{f'}\right)' = \frac{f''f}{f'} + pq\left(\frac{f''}{f'}\right)' - f' = 0 \\ &\Longleftrightarrow p\left(\frac{f''}{f'}\right)' = \frac{f''f}{f'} - f' = 0. \end{aligned}$$

$$\begin{aligned} \frac{f''f}{f'} - f' = 0 &\Longleftrightarrow \left(\frac{f}{f'}\right)' = 0 \\ &\Longleftrightarrow f = C_2 \mathrm{e}^{C_1 z} \quad (C_1 \neq 0, C_2 \neq 0). \end{aligned}$$

Lemma

$$d\bar{\bar{\pi}}_1 \equiv 0 \mod \omega_1 \Longleftrightarrow f = C_2 \mathrm{e}^{C_1 z} \quad (C_1 \neq 0, C_2 \neq 0)$$

Therefor, for $f = C_2 e^{C_1 z}$, $\partial^3 M_1 = \{\bar{\bar{\bar{\pi}}}_1 = \omega_1 = 0\}$,

$$\partial^4 M_1 = \{\bar{\bar{\bar{\pi}}}_1 = \omega_1 = 0\} = \partial^3 M_1.$$

Hence, $F_1 := \{\bar{\bar{\bar{\pi}}}_1 = \omega_1 = 0\}$ satisfies the condition (#).

For M_2, argument is the same.

f-Gordon equation (R, D) is Darboux integrable $\Longleftrightarrow$

$$f \equiv C \quad \text{or} \quad f = C_2 \mathrm{e}^{C_1 z} \quad (C, C_1 \neq 0, C_2 \neq 0).$$

Moreover, for $f \equiv C(\ s = C)$, by

$$\begin{cases} \bar{x} = x \\ \bar{y} = y \\ \bar{z} = z - Cxy \end{cases}$$

we have

$$\bar{s} := \bar{z}_{\bar{x}\bar{y}} = 0.$$

For $f = C_2 e^{C_1 z}$ ($s = C_2 e^{C_1 z}$), by

$$\begin{cases} \bar{x} = x \\ \bar{y} = y \\ \bar{z} = C_1 z + \log |C_1 C_2| \end{cases}$$

we have

$$\bar{s} := \bar{z}_{\bar{x}\bar{y}} = \pm e^{\bar{z}}.$$

The case $C_1 C_2 < 0$,

$$\begin{cases} \bar{\bar{x}} = -\bar{y} \\ \bar{\bar{y}} = \bar{x} \\ \bar{\bar{z}} = \bar{z} \end{cases}$$

$$\bar{\bar{s}} := \bar{\bar{z}}_{\bar{\bar{x}}\bar{\bar{y}}} = e^{\bar{\bar{z}}}.$$

Theorem(S-Yoshimoto)
For f-Gordon equation $(R = \{s - f(z) = 0\}, D)$,

(R, D) is Darboux integrable $\iff f(z) = C$ or $f = C_2 \mathrm{e}^{C_1 z}$ $(C, C_1 \neq 0, C_2 \neq 0)$.

Corollary(S-Yoshimoto)
For a PDE (R, D) which is contact equivalent to f-Gordon equation ,

(R, D) is Darboux integrable $\iff$ the PDE is contact equivalent to $f(z) = 0$ or $f = \mathrm{e}^z$

Corollary(S-Yoshimoto)
Monge characteristic systems $M_i \, (i = 1, 2)$ are regular.

Thank you for your attention !!