微分方程式の幾何学と分類問題

by

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- Differential Equation
- Differential geometry (Differential systems)
- Lie algebra

R : smooth manifold

$D \subset TR$: subbundle

(R, D) : **differential system**, distribution or Pfaffian system

rank $D :=$ the rank of D as a vector bundle

- (local) isomorphism ϕ :

  diffeomorphism $\phi : (R_1, D_1) \to (R_2, D_2) \quad , \quad \phi_*(D_1) = D_2$

Example (Contact manifold: (J, C))

$J := \mathbb{R}^{2n+1} : (x_1, \dots, x_n, y, p_1, \dots p_n)$
Put $\theta := dy - \Sigma_i p_i dx$
$C := \{\theta = 0\} = \{X \in TJ \mid \theta(X) = 0\}$,

Example (The canonical contact system on k-jet space of n independent and m dependent variables: $(J^k(\mathbb{R}^n, \mathbb{R}^m), C^k)$)

$$J^k(\mathbb{R}^n, \mathbb{R}^m) : (x_i, y^\alpha, p_I^\alpha) \quad (1 \leq |I| \leq k) \ ,$$
$$C^k = \{\varpi_I^\alpha = 0 \quad (0 \leq |I| \leq k-1, \ 1 \leq \alpha \leq m)\},$$

where I is a multi-index,

$$\varpi_0^\alpha = dy^\alpha - \textstyle\sum_{i=1}^n p_i^\alpha dx_i \ , \ \varpi_I^\alpha = dp_I^\alpha - \sum_{i=1}^n p_{Ii}^\alpha dx_i.$$

- **Derived System**

$$\partial D := D + [D, D]$$

$$(\partial D \ : \ \partial \mathcal{D} := \mathcal{D} + [\mathcal{D}, \mathcal{D}] \text{ where } \mathcal{D} = \Gamma(D))$$

- The derived system of D is not always a subbundle of TR.
- $D = \partial D \iff D$ is completely integrable.

***i*-th Derived System**: $\partial^i D := \partial(\partial^{i-1} D)$

- Assume $\partial^i D$ are subbundles $(\forall i)$
- $\exists i_0$ s.t.

$$D \subset \cdots \subset \partial^{i_0 - 1} D \subset \partial^{i_0} D = \partial^{i_0+1} D = \cdots \subset TR$$

then, $\partial^{i_0} D$ is the smallest completely integrable subbundle which contains D

- **Cauchy characteristic system** $Ch(D)$ of D

$$Ch(D)(x) := \{X(x) \in D(x) \mid X \in \mathcal{D}\ ,\ [X,Y] \in \mathcal{D}\ (\forall Y \in \mathcal{D})\}$$

where $\mathcal{D} := \Gamma(D)$.

$Ch(D)$:constant rank $\Rightarrow$ $Ch(D)$ is completely integrable

Remark

The Cauchy characteristic system is the biggest completely integrable subbundle contained in D

Example(3-dim contact distribution)
For $(\mathbb{R}^3, D)$

$$D = \{dz - ydx = 0\} = < \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} >$$

Put $\theta := dz - ydx$,

$$d\theta = -dy \wedge dx \neq 0 \quad \mod \theta$$

Therefore, by using

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])$$

we obtain

$$\partial D = T\mathbb{R}^3 \, , \; Ch(D) = \{0\}$$

Definition
For a diffferential sysytem (R, D), a submanifold S is called an **integral manifold** of (R, D), if $TS \subset D$.

Example (Integral manifolds of Contact manifold (J, C))

$J := \mathbb{R}^{2n+1} : (x_1, \dots, x_n, y, p_1, \dots p_n)$
Put $\theta := dy - \Sigma_i p_i dx_i$
$C := \{\theta = 0\} = \{X \in TJ \mid \theta(X) = 0\}$,

Assume that S is a n-dimensional integral manifold of (J, C) with independence condition $dx_1 \wedge \cdots \wedge dx_n|_S \neq 0$. (i.e. S is a submanifold of (J, C) which satisfies $TS \subset C$.)

Then, the n-dim integral manifold S

$S = (x_1, \ldots, x_n, y(x_1, \ldots, x_n), p_1(x_1, \ldots, x_n), \ldots, p_n(x_1, \ldots, x_n))$

of (J, C) satisfies $\iota^*(\theta) = 0$, where $\iota : S \to J$ is the inclusion.

So,

$$
\begin{aligned}
\iota^*(\theta) &= \iota^*(dy - \Sigma_i p_i dx_i) \\
&= (\Sigma_i \frac{\partial y}{\partial x_i} dx_i) - \Sigma_i p_i dx_i \\
&= \Sigma_i (\frac{\partial y}{\partial x_i} - p_i) dx_i \\
&= 0.
\end{aligned}
$$

Hence, we get

$$\frac{\partial y}{\partial x_i}(x_1, \ldots, x_n) = p_i(x_1, \ldots, x_n) \quad (\forall i)$$

Conversely, for any function $y(x_1, \ldots, x_n)$,

$$S = \left(x_1, \ldots, x_n, y(x_1, \ldots, x_n), \frac{\partial y}{\partial x_1}(x_1, \ldots, x_n), \ldots, \frac{\partial y}{\partial x_n}(x_1, \ldots, x_n)\right)$$

is an n-dim integral manifold of (J, C).

So,

$$\{\text{functions } y(x_1, \ldots, x_n)\} = \{\text{integral manifolds } S \text{ of } (J, C)\}$$

Now, we consider a 1st order PDE of n independent and 1dependent variables.

$$F(x_1, \dots, x_n, y, p_1, \dots p_n) = 0$$

satisfying

$$(F_{p_1}, \dots, F_{p_n}) \neq 0$$

We consider a submanifold

$$R := \{F = 0\} \subset J$$

and restrict the contact system C to R. (that is, $D := C|_R \subset TR$)

Then, by the same argument with the above,

$$\{\text{solutions of the PDE} : F = 0\}$$

$$\Longleftrightarrow$$

$$\{\text{integral manifolds of } (R, D) \text{ with independence condition } dx \wedge dy\}.$$

Hence,

"Geometry of 1st order n independent and 1dependent variables PDE"

=

"Submanifold theory in contact manifolds"

In general,

"Geometry of PDE"

=

"Submanifold theory in higher order contact manifolds"

(Here, higher order contact manifold means jet space.)

Geometry of 2nd order 2 independent and 1 dependent variables single PDE was well studied by Lie, Darboux, Goursat, Monge, Cartan, Tresse, etc. around 1900.

Example

$r - t = 0$ (wave equation, hyperbolic)
$r - q = 0$ (heat equation, parabolic)
$r + t = 0$ (Laplace's equation, elliptic)

where, x, y, u, p, q, r, s, t are the classical notations

Example (hyperbolic)

$$\frac{\partial^2 u}{\partial x \partial y} = 0$$

$$R := \{s = 0\} \subset J^2(\mathbb{R}^2, \mathbb{R}) \qquad D = C^2|_\Sigma$$

$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$

$$\begin{aligned} d\varpi_0 &\equiv dx \wedge \varpi_1 + dy \wedge \varpi_2 \qquad \mod \ \varpi_0, \\ d\varpi_1 &\equiv dx \wedge dr \qquad \mod \ \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv dy \wedge dt \qquad \mod \ \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

where

$\{\varpi_0, \varpi_1, \varpi_2, dx, dy, dr, dt\}$:coframe on Σ.

Example (parabolic)

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$

$$\begin{aligned} d\varpi_0 &\equiv dx \wedge \varpi_1 + dy \wedge \varpi_2 \qquad \mod \ \varpi_0, \\ d\varpi_1 &\equiv \qquad dy \wedge ds \qquad \mod \ \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv dx \wedge ds + dy \wedge dt \qquad \mod \ \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

where

$\{\varpi_0, \varpi_1, \varpi_2, dx, dy, ds, dt\}$:coframe on Σ.

Example (elliptic)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$

$$\begin{aligned} d\varpi_0 &\equiv dx \wedge \varpi_1 + dy \wedge \varpi_2 \qquad \mod \ \varpi_0, \\ d\varpi_1 &\equiv dx \wedge dr + dy \wedge ds \qquad \mod \ \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv dx \wedge ds - dy \wedge dr \qquad \mod \ \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

where

$\{\varpi_0, \varpi_1, \varpi_2, dx, dy, dr, ds\}$:coframe on Σ.

After that, the theory is developed by
Bryant, Chern, Gardner, Goldschmidt, and Griffiths, etc. ("MSRI group")
and Tanaka, Morimoto and Yamaguchi, etc. ("Tanaka school")

Motivation

We would like to extend the theory for 2nd order to higher order.

Tomorrow's goal is to give a rough classification for the set of all 3rd order 2 relations PDEs from the view point of Cartan-Kahler theorem and to give characterizations for some classes in the set.

Pfaff-Darboux type Theorem for higher order contact manifolds

M : a manifold of dimension $m+n$

$$J(M,n) = \bigcup_{x\in M} J_x, \quad J_x = \mathbf{Gr}(T_xM, n)$$

We define **Canonical System** C on $J(M,n)$:

$\forall u \in J(M,n)$

$$\begin{array}{ccc} T_u(J(M,n)) & \xrightarrow{\pi_*} & T_xM \\ & & \cup \\ \hat{C}(u) := \pi_*^{-1}(u) & \to & u \end{array}$$

$m = 1 \Rightarrow$ **Contact Manifold**

$\varphi : M \to \hat{M}$:diffeomorphism $\Rightarrow \varphi_* : (J(M,n), C) \to (J(\hat{M},n), \hat{C})$

Theorem (Bäcklund)

$M, \hat{M}$:$m+n-$dim manifold($m \geq 2$).

$\Phi : (J(M,n), C) \to (J(\hat{M},n), \hat{C})$:isomorphism

$\Rightarrow \exists^1 \varphi : M \to \hat{M}$ *such that* $\Phi = \varphi_*$

For $(J(M,n), C)$ $(m \geq 2)$, $F =$Ker π_* is called **Covariant System**.

2nd Order Jet spaces(m=1)

(J, C):**Contact Manifold** $\Rightarrow$ $(L(J), E)$ **Lagrange-Grassmann Bundle**

$$L(J) = \bigcup_{u \in J} L_u \xrightarrow{\pi} J$$

$L_u = \{$**Legendrian subspaces** of $(C(u), d\varpi)\}$
where, $C = \{\varpi = 0\}$. $\forall v \in L(J)$

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J)$$

Ker $\pi_* = Ch(\partial E) \Rightarrow$ **Bäcklund Theorem** for $(L(J), E)$

We define $J^2 := L(J),\ C^2 := E$

2nd Order Jet spaces($m \geq 2$)

$J^2 \subset J(J(M,n),n)$ is defined by

$$J^2 = \{n\text{−}dim.\,\textbf{integral elements}\,of\ (J(M,n),C),\ transversal\,to\,F\}.$$

C^2: Restriction **Canonical System** on $J(J(M,n),n)$ to J^2.

Let (R,D) be a differential system expressed by

$$D = \{\varpi_1 = \cdots = \varpi_s = 0\}.$$

For $x \in R$, $E \subset T_xR$ is an n-dimensional **integral element** of D, if E is an n dimensional subspace in TR such that

$$\varpi_1|_E = \cdots = \varpi_s|_E = d\varpi_1|_E = \cdots = d\varpi_s|_E = 0$$

Namely, integral elements are candidates for the tangent space of integral manifolds of D.

Higher Order Jet spaces

Inductively, we define $J^{k+1} \subset J(J^k, n)$ for $k \geq 2$ by

$$J^{k+1} = \{n\text{-dim } \textbf{integral elements} \text{ of } (J^k, C^k), \quad \text{transversal to Ker } (\pi^k_{k-1})_*\}$$

C^{k+1}:**Canonical System** on J^{k+1} where ,Ker $(\pi^k_{k-1})_* = Ch(\partial C^k)$.

$$\begin{array}{ccccccc} C^k & \subset \cdots \subset & \partial^{k-2} C^k & \subset & \partial^{k-1} C^k & \subset & T(J^k) \\ \cup & & \cup & & \cup & & \\ Ch(C^k) \subset Ch(\partial C^k) & \subset \cdots \subset & Ch(\partial^{k-1} C^k) & \subset & F & & \end{array}$$

Transversality condition:

$C^k \cap F = Ch(\partial C^k) (m \geq 2)$

$C^k \cap Ch(\partial^{k-1} C^k) = Ch(\partial C^k) (m = 1)$

[Yamaguchi (1982,1983)]

Contact isomorphisms

$m = 1$

$$\{\text{Diffeo.}\} \subset \{\text{Contact iso.}\} = \{\text{Contact iso. for higher order}\}$$

$m > 1$

$$\{\text{Diffeo.}\} = \{\text{Contact iso.}\} = \{\text{Contact iso. for higher order}\}$$

Weak Derived system, Symbol algebra

k-th weak higher derived system $\partial^{(k)}\mathcal{D}$ is defined by

$$\partial^{(1)}\mathcal{D} = \partial\mathcal{D}\ ,\ \partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}]$$

where $\mathcal{D} = \Gamma(D)$.

D is **weakly regular** $\iff$ $\partial^{(i)}D$ is subbundle $(\forall i)$.

Proposition(Tanaka)
For weakly regular differential system D;

$$(S1)\ \exists\mu\ \text{ s.t. }\ D^{-1} \subset D^{-2} \subset \cdots \subset D^{-k} \subset \cdots$$
$$\cdots \subset D^{-(\mu-1)} \subset D^{-\mu} = D^{-(\mu+1)} = \cdots$$
$$(S2)\ [\mathcal{D}^p, \mathcal{D}^q] \subset D^{p+q} \qquad \forall p, q < 0$$
$$i.e.\ [X, Y] \in \mathcal{D}^{p+q}\ , \quad X \in \mathcal{D}^p, Y \in \mathcal{D}^q \quad \forall p, q < 0$$

where $D^{-1} := D,\ D^{-k} := \partial^{(k-1)}D\ (k \geq 2)$

Symbol Algebra of weakly regular differential system

(R, D):weakly regular differential system ,

$T(R) = D^{-\mu} \supset D^{-(\mu-1)} \supset \cdots \supset D^{-1} = D$

$\forall x \in R$, $\mathfrak{g}_{-1}(x) := D^{-1}(x) = D(x), \mathfrak{g}_p(x) := D^p(x)/D^{p+1}(x)$

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

dim $\mathfrak{m}(x)$=dim R.

For $X \in \mathfrak{g}_p(x)$, $Y \in \mathfrak{g}_q(x)$, we define $[X, Y] \in \mathfrak{g}_{p+q}(x)$ by :

let $\tilde{X} \in \mathcal{D}^p$, $\tilde{Y} \in \mathcal{D}^q$ be extensions ($\tilde{X}_x = X$, $\tilde{Y}_x = Y$), then $[\tilde{X}, \tilde{Y}] \in \mathcal{D}^{p+q}$ $[X, Y] := [\tilde{X}, \tilde{Y}]_x \in \mathfrak{g}_{p+q}(x)$ is not depend on the extensions.

$(\mathfrak{m}(x), [\ ])$ is called Symbol Algebra of (R, D) at x

- $(\mathfrak{m}(x), [\ ])$ is nilpotent graded Lie algebra
- $[\mathfrak{g}_{-1}, \mathfrak{g}_p] = \mathfrak{g}_{-1+p}$ (generating condition) holds.

Conversely, for a graded Lie algebra $\mathfrak{m}$ which satisfies the above condition (FGLA), $\exists(R, D)$ s.t. symbol algebra of (R, D) is isomorphic to the $\mathfrak{m}$ at any points.

Examples(Martinet distribution)
$(\mathbb{R}^3, D)$

$$D = \{dz - y^2 dx = 0\} =< \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial z} >$$

$$\begin{aligned} \partial D &= D && \text{on } \{y = 0\} \\ \partial D &= T\mathbb{R}^3 && \text{on } \{y \neq 0\} \end{aligned}$$

Not weak regular. Not defined symbol.

Example(contact distribution)

$(\mathbb{R}^3, D)$

$$D = \{dz - ydx = 0\} =< \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} >$$

$$\partial D = T\mathbb{R}^3$$

Symbol algebra is isomorphic to Heisenberg Lie alg.:

$$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$$

$$\mathfrak{g}_{-1} =< \{X, Y\} > \ , \ \mathfrak{g}_{-2} =< \{Z\} >$$

$$Z = [Y, X]$$

Remark

- Prolongation
- Finite, Infinite
- Simple Lie alg.

Finsler Geometry and Contact Geometry

An (I, J, K)-generalized Finsler structure on a 3-manifold is a generalization of a Finslerian structure, introduced in order to separate and clarify the local and global aspects in Finsler geometry making use of the Cartan's method of exterior differential systems.

In this talk, we introduce that there is a close relation between $(I, J, 1)$-generalized Finsler structures and a class of contact circles, namely the so-called Cartan structures .

This correspondence allows us to determine the topology of 3-manifolds that admit $(I, J, 1)$-generalized Finsler structures and to single out classes of $(I, J, 1)$-generalized Finsler structures induced by standard Cartan structures.

Definition of (I, J, K)-generalized Finsler structure

(classical) Finsler Manifold : (M, F)

R: C^∞ manifold (dim $R = n$)
$F : TM \to [0, \infty)$: Finsler metric

(1) F is smooth on $TM \setminus \{0\}$ and continuous at zero section.

(2) $F(x, \lambda v) = \lambda F(x, v) \qquad (\lambda > 0)$
(3)

$$g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is positive definite for all $v = (x, y) \in TM \setminus \{0\}$

In this talk, we consider only $n = 2$ case.

Theorem(Chern)
(M^2, F) : classical Finsler manifold $\Rightarrow$
There exists a unique coframe $\omega = \{\omega_1, \omega_2, \omega_3\}$ on Σ_F s.t.

$$d\omega^1 = -I\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3$$
$$d\omega^2 = -\omega^1 \wedge \omega^3$$
$$d\omega^3 = K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3$$

where

$$\Sigma_F := \{v \in TM : F(v) = 1\} \subset TM$$

is indicatrix.

- Chern proved the existence of the coframe for $n \geq 2$.

(M^2, g):Riemannian manifold $\Rightarrow$

Σ : indicatrix

There exists a unique coframe $\omega = \{\omega_1, \omega_2, \omega_3\}$ on Σ

$$d\omega^1 = \omega^2 \wedge \omega^3$$
$$d\omega^2 = -\omega^1 \wedge \omega^3$$
$$d\omega^3 = K\omega^1 \wedge \omega^2$$

K : Gauss curvature

Definition(Bryant 1996)

(Σ, ω) : (I, J, K)-generalized Finsler structure(GFS)

Σ : 3-dim manifold

$\omega = \{\omega_1, \omega_2, \omega_3\}$: coframe on Σ

$$
\begin{aligned}
d\omega^1 &= -I\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 \\
d\omega^2 &= -\omega^1 \wedge \omega^3 \\
d\omega^3 &= K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3
\end{aligned}
$$

where I, J, K are functions on Σ.

Remark

- Bryant defined the GFSs for any $n \geq 2$.

- For (I, J, K)-generalized Finsler structure, if $I = 0$, then $J = 0$ by Bianchi identities. Hence, $(0, 0, K)$-generalized Finsler structure.

(Roughly speaking, $(0, 0, K)$-generalized Finsler structure is Riemann structure)

Theorem (Bryant 1996) GFS:(Σ, ω) to be realizable as a classical Finsler structure on a surface

$\Longleftrightarrow$

1. the leaves of the foliation $\{\omega^1 = 0,\ \omega^2 = 0\}$ are compact;
2. it is amenable, i.e. the space of leaves of the foliation $\{\omega^1 = 0,\ \omega^2 = 0\}$ is a differentiable manifold M;
3. the canonical immersion $\iota : \Sigma \to TM$, given by $\iota(u) = \pi_{*,u}(\hat{e}_2)$, is one-to-one on each π-fiber Σ_x,

where we denote by $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ the dual frame of the coframing $(\omega^1, \omega^2, \omega^3)$.

Remark

- The condition (2) of Bryant's Theorem guarantees the existence of the base surface M.

- (1) guarantees the each fiber $\Sigma \to M$ is diffeomorphic to S^1

- (3) guarantees the each fiber $\Sigma \to M$ which is diffeomorphic to S^1 have no selfintersections

Definition of $\mathcal{K}$-Cartan structure on Σ^3

Σ : C^∞ manifold ($\dim \Sigma = 3$)

Definition

A *contact form* on a 3-manifold Σ is a 1-form α such that $\alpha \wedge d\alpha \neq 0$, that is, is a volume form.

Definition

A 3-manifold Σ is said to admit a *contact circle* if it admits a pair of contact forms (α^1, α^2) such that for any $(\lambda_1, \lambda_2) \in S^1$, the linear combination $\lambda_1\alpha^1 + \lambda_2\alpha^2$ is also a contact form.

Definition

A contact circle (α^1, α^2) is called a *taut contact circle* if the contact forms $\lambda_1\alpha^1 + \lambda_2\alpha^2$ define the same volume form for all $(\lambda_1, \lambda_2) \in S^1$.

That is, the contact circle (α^1, α^2) satisfies

$$\begin{aligned} \alpha^1 \wedge d\alpha^1 &= \alpha^2 \wedge d\alpha^2 \neq 0 \\ \alpha^1 \wedge d\alpha^2 + \alpha^2 \wedge d\alpha^1 &= 0 \end{aligned}$$

Definition

The contact circle (α^1, α^2) is called a *Cartan structure* on the 3-manifold Σ if the following conditions are satisfied

$$\begin{aligned} &\alpha^1 \wedge d\alpha^1 = \alpha^2 \wedge d\alpha^2 \neq 0 \\ &\alpha^1 \wedge d\alpha^2 = 0, \quad \alpha^2 \wedge d\alpha^1 = 0. \end{aligned} \tag{1}$$

Definition

The Cartan structure (α^1, α^2) is called a $\mathcal{K}$-*Cartan structure* if the unique form η(this always exists) satisfies the structure equation

$$d\alpha^1 = \alpha^2 \wedge \eta$$
$$d\alpha^2 = \eta \wedge \alpha^1$$
$$d\eta = \mathcal{K}\alpha^1 \wedge \alpha^2.$$

Remark By the definitions,

- A $\mathcal{K}$-Cartan structure is a taut contact circle.

- A $\mathcal{K}$-Cartan structure is a $(0, 0, \mathcal{K})$-generalized Finsler structure.

One of the main results of this theory is the following:

Theorem(Geiges and Gonzalo 1995)
Let Σ be a closed 3-manifold. Then Σ admits a taut contact circle if and only if Σ is diffeomorphic to a quotient of the Lie group G under a discrete subgroup Γ of G, acting by left multiplication, where G is one of the following:

1. $S^3 = SU(2)$, the universal cover of $SO(3)$,
2. $\widetilde{SL}_2$, the universal cover of $PSL_2(\mathbb{R})$,
3. $\widetilde{E}_2$, the universal cover of the Euclidean group, i.e. orientation preserving isometries of $\mathbb{R}^2$.

Remark

- For the Lie group G in Theorem, all discrete cocompact subgroups Γ of G are known.

Proposition(Pitis-Sabau-S) Let (Σ, ω) be an (I, J, K)-generalized Finsler structure on a closed 3-manifold Σ, where $\omega = (\omega^1, \omega^2, \omega^3)$. Then we have

1. (ω^1, ω^2) is a taut contact circle if and only if $I = 0$, i.e. (Σ, ω) is in fact a $\mathcal{K} := K$-Cartan structure;
2. (ω^1, ω^3) is a taut contact circle if and only if $K = 1$, i.e. (Σ, ω) is an $(I, J, 1)$-generalized Finsler structure. This taut contact circle is actually a $\mathcal{K}$-Cartan structure on Σ;
3. (ω^2, ω^3) is a taut contact circle if and only if $K = 1$ and $J = 0$. Moreover, $I = 0$ and (Σ, ω) is a 1-Cartan structure.

Remark

- The case of (1) and (3) are "Riemann" case. So we consider the non-trivial case (2) $(I, J, 1)$-generalized Finsler structure.

Theorem(Pitis-Sabau-S)

Let Σ be a closed 3-manifold. Then Σ admits an $(I, J, 1)$-generalized Finsler structure $\iff$ it is diffeomorphic to a quotient of the Lie group G under a discrete subgroup Γ of G, where G is one of the following:

1. $S^3 = SU(2)$, the universal cover of $SO(3)$,
2. $\widetilde{SL}_2$, the universal cover of $PSL_2(\mathbb{R})$,
3. $\widetilde{E}_2$, the universal cover of the Euclidean group, i.e. orientation preserving isometries of $\mathbb{R}^2$.

Sketch of proof

($\Rightarrow$) By Proposition and Thoerem(Geiges and Gonzalo 1995).
($\Leftarrow$) This direction is essential and difficult part.

1. Take a $\mathcal{K}$-Cartan structure $(\alpha_1, \alpha_2, \eta)$ on Σ/Γ.
2. Write a condition of a 1-form φ which change the $\mathcal{K}$-Cartan structure to $(I, J, 1)$-generalized Finsler structure $\omega = (\omega_1, \omega_2, \omega_3)$.
$$\omega^1 = \alpha^1$$
$$\omega^2 = \varphi - \eta$$
$$\omega^3 = \alpha^2$$
3. Rewrite the condition by the theory of the Liouville-Cartan sturucture.
4. Construct a 1-form satisfying this condition.

$\square$

Thank you for your attention !!