Stability for C^∞ maps of manifolds with submanifolds and their singularities

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接触構造，特異点，微分方程式及びその周辺

旭川市ときわ市民ホール「多目的ホール」

Aim: To characterize "Stability" of C^∞ map germs $(R^n_m, 0) \to (\mathbb{R}^p, 0)$.
To introduce Thom-Bordman type characterization of singular points of map germs $(R^n_m, 0) \to (\mathbb{R}^p, 0)$.

Contents

§1 Map germs $(R^n_m, 0) \to (\mathbb{R}^p, 0)$
§2 Example of stable singularities
§3 Thom-Bordman type characterization

In this talk, all manifolds N and maps $N \to \mathbb{R}^p$ are class C^∞.

Denote $(R^n_m, 0)$ the set germ at $0 \in \mathbb{R}^n$ of the pair

$$R^n_m := (\mathbb{R}^n, \mathbb{R}^m \times \{0\}),\ \ n > m$$

where $\mathbb{R}^n$ equipped $(x_1, \ldots, x_m, y_1, \ldots, y_{n-m})$ coordinates and $\mathbb{R}^m \times \{0\}$ is the $\{y_1 = \cdots = y_{n-m} = 0\}$-plane. This corresponds to a submanifold of our manifold.

! For a map germ $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$,
$Df_0 \neq D(f|_{\mathbb{R}^m \times \{0\}})$ and $j^k f(0) \neq j^k(f|_{\mathbb{R}^m \times \{0\}})(0)$.

In the following assume $n \geq p$.

f and $g\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ are *Relative-$\mathcal{A}$-equivalent* (*Relative-equivalent* for short)

$\overset{\text{def}}{\Leftrightarrow}$ $^\exists$diffeo. germs

$s\colon (R^n_m, 0) \to (R^n_m, 0)$ which preserves $\mathbb{R}^m \times \{0\}$,

i.e. $s = (s_1, \ldots, s_m, s_{m+1}, \ldots, s_n)$ with $s_{m+i} = \alpha_i \widetilde{s_{m+i}}$,

where $\alpha_i \in m_{n-m}.\mathcal{E}_n = < y_1, \ldots, y_{n-m} >_{\mathcal{E}_n} \subset m_n$,

$t\colon (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$

s.t. they make the following diagram commute:

$$\begin{array}{ccc} (R^n_m, 0) & \xrightarrow{f} & (\mathbb{R}^p, 0) \\ s \downarrow & & \downarrow t \\ (R^n_m, 0) & \xrightarrow{g} & (\mathbb{R}^p, 0). \end{array}$$

$$\mathcal{RA} = \left\{ (s,t) \;\middle|\; \begin{array}{r} s\colon (R^n_m, 0) \to (R^n_m, 0) : \text{diffeo. preserving } \mathbb{R}^m \times \{0\} \\ t\colon (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0) : \text{diffeo} \end{array} \right\}$$

! $\mathcal{RA} \subset \mathcal{A} = \left\{ (s,t) \;\middle|\; \begin{array}{r} s\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) : \text{diffeo.} \\ t\colon (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0) : \text{diffeo.} \end{array} \right\}$: subgroup

! Function germs $(X^n, 0) \to (\mathbb{R}, 0)$ of manifolds **with boundary** was studied Arnold ('76) and Siersma ('81), Shcherbak ('91), Tsukada ('96).

! Map germs $(X^2, 0) \to (\mathbb{R}^2, 0)$ of surfaces **with boundary** was studied by Bruce and Giblin ('90).

! Map germs $(X^3, 0) \to (\mathbb{R}^2, 0)$ of 3-manifolds **with boundary**was studied by Shibata('00), Martins and Nabbaro ('13).

! Saeki and Y ('14) studied singular fibers of stable maps $f\colon N^3 \to P^2$ of 3-mfds. **with bdry**. into surfaces without bdry., and obtained a non-trivial $\mathcal{AS}_{\text{pr}}(3,2)$-cobordism invariant of Morse maps $f\colon V^2 \to W^1$ of surfaces with bdry. into $W^1 = \mathbb{R}$ or S^1.

Shcherbak ('91) denote singularities of functions $f\colon (X^n, 0) \to (\mathbb{R}, 0)$ of manifolds **with boundary** ($x = 0$) by the singularities of the ambient space and the singularities of the restriction of the boundary.

1. Singularities with decomposition (A, A). Besides the simple singularities $B_k \equiv B^1_{k-1}$, $C_k \equiv C^{k-1}_1$, $F_4 \equiv F^2_2$ (cf. [5]) there is also an infinite series of equivalence classes ($k \geq 1$).

List 1

Class	Normal Form	m	Decomposition	Duals
F^{2k+1}_{2k+1}	$x^2 + a_1xy^{k+1} + y^{2k+2} + a_2xy^{k+2} + \cdots + a_kxy^{2k}$, $a_1^2 \neq 4$	k	A_{2k+1}, A_{2k+1}	F^{2k+1}_{2k+1}
F^{2k}_{2k}	$x^2 + y^{2k+1} + a_1xy^{k+1} + \cdots + a_{k-1}xy^{k-1}$	$k-1$	A_{2k}, A_{2k}	F^{2k}_{2k}
B^{2k-1}_l $l \geq 2k$	$(x + y^k)^2 + a_1y^{l+1} + \cdots + a_{k-1}y^{k+l-1}$, $a_1 \neq 0$	$k-1$	A_l, A_{2k-1}	C^l_{2k-1}
C^l_{2k-1} $l \geq 2k$	$x^2 + 2xy^k + a_1y^{l+1} + \cdots + a_{k-1}y^{k+l-1}$, $a_1 \neq 0$	$k-1$	A_{2k-1}, A_l	B^{2k-1}_l

"*Boundary Singularities with a simple decomposition*", Journal of Soviet Mathematics July 1992, Volume 60, Issue 5, pp 1681-1693.

$f: (R^n_m, 0) \to (\mathbb{R}^p, 0)$ is *stable*

$\overset{\text{def}}{\Leftrightarrow}$ $^\forall$ representative $f: U \to \mathbb{R}^p$ of f,

$^\exists N(f) \subset C^\infty(U, \mathbb{R}^p)$: an open neighborhood of f

s.t. $^\forall f' \in N(f)$, $^\exists u' \in U$,

germs $(f, 0)$ and (f', u') are Relative-$\mathcal{A}$-equivalent.

! $C^\infty(U, \mathbb{R}^p)$ is equipped with the Whitney C^∞ toplogy.

$f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ is *homotopically stable*
$\overset{\text{def}}{\Leftrightarrow}$ $^\forall$unfolding $\Phi\colon (R^n_m \times \mathbb{R}^k, (0,0)) \to (\mathbb{R}^p, 0)$ of f is locally trivial,

Let $\Phi\colon (R^n_m \times \mathbb{R}^k, (0,0)) \to (\mathbb{R}^p, 0)$ be an unfolding of f.
$\Phi\colon (R^n_m \times \mathbb{R}^k, (0,0)) \to (\mathbb{R}^p, 0)$ is *locally trivial*
$\overset{\text{def}}{\Leftrightarrow}$ $^\exists$diffeo. germs $h\colon (R^n_m \times \mathbb{R}^n, (0,0)) \to (R^n_m \times \mathbb{R}^p, (0,0))$ and
$H\colon (\mathbb{R}^p \times \mathbb{R}^k, (0,0)) \to (\mathbb{R}^p \times \mathbb{R}^k, (0,0))$ s.t. they satisfies:

(1) $h(x,0) = (x,0)$, $h(X,0) = (X,0)$,
(2) h preserves $\mathbb{R}^m \times \{0\}$,
(3) they make the following diagram commutes

$$\begin{array}{ccccc}
(R^n_m \times \mathbb{R}^k, (0,0)) & \xrightarrow{(\Phi,\pi)} & (\mathbb{R}^p \times \mathbb{R}^k, (0,0)) & \xrightarrow{\pi'} & (\mathbb{R}^k, 0) \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \mathrm{id}_{(\mathbb{R}^k,0)}} \\
(R^n_m \times \mathbb{R}^k, (0,0)) & \xrightarrow{(f,\pi)} & (\mathbb{R}^p \times \mathbb{R}^k, (0,0)) & \xrightarrow{\pi'} & (\mathbb{R}^k, 0).
\end{array}$$

For a C^∞ map germ $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$, let
$\theta(f)$: the set of vector fields along f,
$\theta(n,m)$: the set of vector fields on $(R^n_m, 0)$
tangent to $\mathbb{R}^m \times \{0\}$ on $\mathbb{R}^m \times \{0\}$,
$\theta(p)$: the set of vector fields on $(\mathbb{R}^p, 0)$.

! $\theta(f) = \mathcal{E}^p_n$, $\theta(p) = \mathcal{E}^p_p$ and

$$\theta(n,m) = \left\{ \xi_1 \frac{\partial}{\partial x_1} + \cdots + \xi_m \frac{\partial}{\partial x_m} + \alpha_1 \frac{\partial}{\partial y_1} + \cdots + \alpha_{n-m} \frac{\partial}{\partial y_{n-m}} \mid \xi_i \in \mathcal{E}_n, \alpha_j \in m_{n-m} \right\},$$

where $\mathcal{E}_n$ denote the ring consisting of all C^∞ function germs $(\mathbb{R}^n, 0) \to \mathbb{R}$ and $m_{n-m} \subset \mathcal{E}_n$ consisting of all C^∞ function germs $(\mathbb{R}^{n-m}, 0) \to (\mathbb{R}, 0)$.

Then, $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ defines maps
$tf\colon \theta(n,m) \to \theta(f)$ by $\xi \mapsto Tf(\xi)$ and $\omega f\colon \theta(p) \to \theta(f)$ by $\eta \mapsto \eta \circ f$.

A map germ $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ is *strongly infinitesimal stable*
$\overset{\text{def}}{\Leftrightarrow}$ it satisfies $\theta(f) = tf(\theta(n,m)) + \omega f(\theta(p))$
$\overset{\text{iff}}{\Leftrightarrow}$ $\mathcal{E}^p_n = \mathcal{E}_n.\{f_{x_1}, \ldots, f_{x_m}, \alpha_1 f_{y_1}, \ldots \alpha_{n-m} f_{y_{n-m}}\} + \omega f(\mathcal{E}^p_p)$, $(\alpha_i \in m_{n-m})$.

Prop

Let $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ be a C^∞ map germ. Then, the following six conditions are equivalent:

(1) f is stable,

(2) f is strongly infinitesimal stable,

(3) f is homotopically stable,

(3) $\mathcal{E}^p_n = \mathcal{E}_n.\{f_{x_1}, \dots, \alpha_1 f_{y_1}, \dots \alpha_{n-m} f_{y_{n-m}}\} + \omega f(\mathcal{E}^p_p) + f^* m_p.\mathcal{E}^p_n$,

(4) $\mathcal{E}^p_n = \mathcal{E}_n.\{f_{x_1}, \dots, \alpha_1 f_{y_1}, \dots \alpha_{n-m} f_{y_{n-m}}\} + \omega f(\mathcal{E}^p_p) + f^* m_p.\mathcal{E}^p_n + m^{p+1}_n.\mathcal{E}^p_n$,

(5) For any representation $\widetilde{f}$ of f and any $\mathcal{RA}^p$-orbit $\mathcal{RA}^p(z)$, $j^p\widetilde{f}$ is transverse to $\{0\} \times \mathbb{R}^p \times \mathcal{RA}^p(z)$,

(6) For any representation $\widetilde{f}$ of f and any $\mathcal{RK}^p$-orbit $\mathcal{RK}^p(z)$, $j^p\widetilde{f}$ is transverse to $\{0\} \times \mathbb{R}^p \times \mathcal{RK}^p(z)$.

! This prop. is proved as the similarly way of the cases $\mathcal{A}$-eq. & $\mathcal{K}$-eq.

! If $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ is stable, then $f|_{\mathbb{R}^m \times \{0\}}$ is also stable.

Prop

$f, g\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$: stable map germs

f and g are relative-$\mathcal{A}$-equivalent $\overset{\text{iff}}{\Leftrightarrow}$ they are relative-$\mathcal{K}$-equivalent

! Germs f and $g\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ are *relative-$\mathcal{K}$-equivalent*

$\overset{\text{def}}{\Leftrightarrow}$ $^\exists$ a diffeo. germ $s\colon (R^n_m, 0) \to (R^n_m, 0)$ and

$^\exists$ a C^∞ map $M\colon (R^n_m, 0) \to (GL(p, \mathbb{R}), M(0))$

s.t. s preserves $\mathbb{R}^m \times \{0\}$,

i.e. $s = (s_1, \dots, s_n)$ with $s_i = \alpha_i \widetilde{s}_i$ $(i = m+1, \dots, n)$, $\alpha_i \in m_{n-m}.\mathcal{E}_n$

they make the following diagram commute:

$$\begin{array}{ccc} (R^n_m, 0) & \xrightarrow{f} & (\mathbb{R}^p, 0) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle M(x)} \\ (R^n_m, 0) & \xrightarrow{g} & (\mathbb{R}^p, 0). \end{array}$$

! $m^r_n \theta(f) \subset T\mathcal{RK}(f) = tf(m_n \theta(n, m)) + f^* m_p \theta(f)$

$\Rightarrow$ f is r-determined with resp. to $\mathcal{RK}$.

Cf. f is r-determined with resp. to $\mathcal{RK}$

$\overset{\text{def}}{\Leftrightarrow}$ "$j^r f(0) = j^r g(0) \Rightarrow f \sim_{\mathcal{RK}} g$"

Let $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ be a C^∞ map germ satisfying

$$\dim_{\mathbb{R}} \frac{\theta(f)}{tf(\theta(n,m)) + f^* m_p \theta(p)} = k + \ell,$$

and $\psi_1, \ldots, \psi_k \in m_n\theta(f)$, $\mathbf{a}_1, \ldots, \mathbf{a}_\ell \in \mathbb{R}^p$ span the $\mathbb{R}$-vector space via the projection $\theta(f) \to \dfrac{\theta(f)}{tf(\theta(n,m)) + f^* m_p \theta(p)}$. Then, we have the following.

Prop

A germ $F\colon (\mathbb{R}^k \times R^n_m, (0,0)) \to (\mathbb{R}^k \times \mathbb{R}^p, (0,0))$ i.e
$F\colon (R^{n+k}_{m+k}, 0) \to (\mathbb{R}^{p+k}, 0)$ defined by

$$F(\lambda_1, \ldots, \lambda_k, x) = (\lambda_1, \ldots, \lambda_k, f(x) + \sum_{i=1}^{k} \lambda_i \psi_i)$$

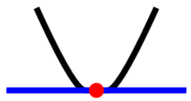
is a stable map germ.

Example Let $f_\pm = x^2 \pm y^2\colon (R^2_1, 0) \to (\mathbb{R}, 0)$. Then,

$$\begin{aligned} \mathcal{E}_2/(tf_\pm(\theta(1,1)) + f_\pm^* m_1 \mathcal{E}_1) &= \mathcal{E}_2/(2x\xi \pm 2y\alpha + (x^2 \pm y^2)\varphi) \\ &\qquad (\xi, \varphi \in \mathcal{E}_2, \alpha \in m_1 =< y >_{\mathcal{E}_2}) \\ &= < 1, y >_{\mathbb{R}} . \end{aligned}$$

Thus, we obtain an $\mathcal{RK}$-2-determined stable germ

$$F_\pm = (a, x^2 \pm y^2 + ay)\colon (R^3_2, 0) \to (\mathbb{R}^2, 0)$$



$F_\pm(S(F_\pm)) \cup$ $F_\pm(S(F_\pm|_{y=0}))$

! $F_\pm$ has FOLD (A_1, $\Sigma^{2,0}$) singularities along $\{(\mp 2y, 0, y)\}$ and $F_\pm|_{\mathbb{R}^2 \times \{0\}}$ has FOLD (A_1, $\Sigma^{1,0}$) singularities along $\{(a, 0, 0)\}$.

A map germ $F_2(a, x, y) = (a, x^2 \pm y^2)$ is **NOT** stable. Note that both F_2 and $F_2|_{\mathbb{R}^2 \times \{0\}}$ has fold singularity along $(a, 0, 0)$.

In particular, a map germ $f\colon (R^{2n+1}_{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ defined by

$$f = (a_1, \ldots, a_n, x^2 \pm y_1^2 \pm \cdots \pm y_n^2 + a_1 y_1 + \cdots + a_n y)$$

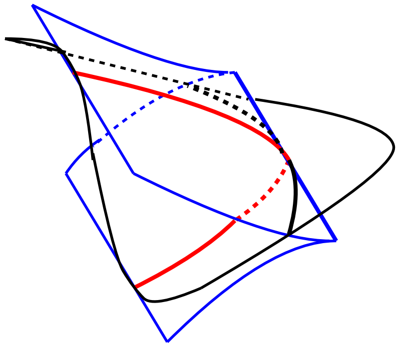
is a stable map germ s.t. both f and $f|_{\mathbb{R}^{n+1}\times\{0\}}$ have FOLD (A_1) singularities.

Furthermore, a map germ $f\colon (R^{n+nk+k}_{n+nk}, 0) \to (\mathbb{R}^{n+nk}, 0)$ defined by

$$\begin{aligned} f = (a_1, \ldots, a_{n-1}, a^1_1, \ldots, a^1_n, \ldots, a^k_1, \ldots, a^k_n \quad , \quad & \pm x^{n+1} \pm y_1^2 \pm \cdots \pm y_k^2 \\ & + a_1 x + \cdots + a_{n-1} x^{n-1} \\ & + a^1_1 y_1 + \cdots + a^1_n x^{n-1} y_1 \\ & \vdots \\ & + a^k_1 y_k + \cdots + a^k_n x^{n-1} y_k) \end{aligned}$$

is a stable map germ s.t. both f and $f|_{\mathbb{R}^{n+nk}\times\{0\}}$ have A_n singularities.

Example $(R^4_3, 0) \to (\mathbb{R}^3, 0)$
A map germ $f_\pm(a,b,x,y) = (a, b, xy \pm x^3 + ax + by)$ is stable.



$f_\pm(S(f_\pm)) \cup f_\pm(S(f_\pm|_{y=0}))$

! $f_\pm$ has FOLD (A_1, $\Sigma^{2,0}$) singularities and $f_\pm|_{\mathbb{R}^3 \times \{0\}}$ has CUSP (A_2, $\Sigma^{1,1,0}$) singularities.

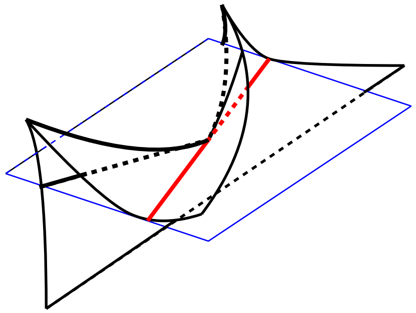
In particular, the map germ $f\colon (R^{n+2k+2}_{n+k+1}, 0) \to (\mathbb{R}^{n+k+1}, 0)$ defined by

$$f = (a_1, \ldots, a_n, b_1, \ldots, b_k \quad , \quad xy \pm x^{n+1} \pm z_1^2 \pm \cdots \pm z_k^2 + a_1 x + \cdots$$
$$+ a_{n-1}x^{n-1} + a_n y + b_1 z_1 + \cdots + b_k z_k)$$

is a stable map germ s.t. f is A_1 singularities and $f|_{\mathbb{R}^{n+k+1} \times \{0\}}$ is A_n singularities.

Example $(R^4_3, 0) \to (\mathbb{R}^3, 0)$

A map germ $f_\pm(a,b,x,y) = (a, b, x^2 \pm y^3 + ay + by^2)$ is stable.



$f_\pm(S(f_\pm)) \cup f_\pm(S(f_\pm|_{y=0}))$

! $f_\pm$ has CUSP (A_2, $\Sigma^{2,1,0}$) singularities and $f_\pm|_{\mathbb{R}^3\times\{0\}}$ has FOLD (A_1, $\Sigma^{1,0}$) singularities.

In particular, the map germ $f\colon (R^{n+2k+2}_{n+k+1}, 0) \to (\mathbb{R}^{n+k+1}, 0)$ defined by

$$f = (a_1, \dots, a_n, b_1, \dots, b_k \quad , \quad x^2 \pm y^{n+1} \pm z_1^2 \pm \cdots \pm z_k^2 + a_1 y + \cdots + a_n y^n + b_1 z_1 + \cdots + b_k z_k)$$

is a stable map germ s.t. f is A_n singularities and $f|_{\mathbb{R}^{n+k+1}\times\{0\}}$ is A_1 singularities.

Let Σ^I and Σ^J be Thom-Boardman symbols for $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ and $(\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ respectively.

Definition

A map germ $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ is *Σ^I_J type*

$\overset{\text{def}}{\Leftrightarrow}$ f is Σ^I tyep if we ignore the submfd $\mathbb{R}^m \times \{0\}$, and $f|_{\mathbb{R}^m \times \{0\}}$ is Σ^J tyep.

! Let $\Sigma^I_J = \Sigma^{n-p+1,1,\ldots,1,0}_{m-p+1,0}$ or $\Sigma^{n-p+1,0}_{m-p+1,1,\ldots,1,0}$, $\Sigma^{n-p+1,1,\ldots,1,0}_{m-p+1,1,\ldots,1,0}$ (the same number of 1s). Then,

A Σ^I_J type map germ $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ is stable

$\overset{\text{iff}}{\Leftrightarrow}$ $0 \in \Sigma^I(f)$ and $\Sigma^I(f) \pitchfork \mathbb{R}^m \times \{0\}$ at 0.

Q When a map germ $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ admits a stable singularities of type Σ^I_J?

§Thom-Boardman type symbol

A C^∞ map $f\colon (R^n_m, 0) \to (R^p, 0)$ is *relative* if $f(\mathbb{R}^m \times \{0\}) \subset \mathbb{R}^q \times \{0\}$. Then, denote by $f\colon (R^n_m, 0) \to (R^p_q, 0)$ the relative map.

$C^\infty(n, m; p, q) := \{f\colon (R^n_m, 0) \to (R^p_q, 0)\}$.
! $C^\infty(n, m; p, q)$ is equipped with the induced topology from the Whitney C^∞ topology on $C^\infty(n, p)$. Then, $C^\infty(n, m; p, q)$ is a Baire space. Thus, a countable intersection of open dense subsets of $C^\infty(n, m; p, q)$ is dense in $C^\infty(n, m; p, q)$.
! If $q = p$, then $C^\infty(n, m; p, p) = C^\infty(n, p)$.

Assume $k \geq 0$.
$J^k(n, m; p, q) := \{j^k f(0) \in J^k(n, p) | f\colon (R^n_m, 0) \to (R^p_q, 0)\}$
Then, $J^k(n, m; p, q)$ is a fibration over $(\mathbb{R}^m \times \{0\}) \times (\mathbb{R}^q \times \{0\})$.

Relative Transversality Theorem: germ version (G.Ishikawa)

For a given countable family T of $J^k(n,m;p,q)$,
$^{\exists}R \subset C^{\infty}(n,m;p,q)$: a residual subset
s.t. $^{\forall}f \in R$ satisfies that
$j^k f|_{\mathbb{R}^m \times \{0\}} \colon \mathbb{R}^m \times \{0\} \to J^k(n,m;p,q)$ is transverse to T

Let us apply Relative transversality theorem to the case $q = p$. Then, Relative Transversality theorem shows that for a given triple integers n, m, p and a type of singularity $\mathcal{S}$ if stable maps $f \colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ admit singularities of types $\mathcal{S}$.

Let $\Sigma^I \subset J^k(n,p)$ and $\Sigma^J \subset J^k(m,p)$ be Thom-Boardman submfds, where $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$. Put

$$\Sigma^I_J := \Sigma^I \cap \pi^{-1}(\Sigma^J) \subset J^k(n,p),$$

where $\pi\colon J^k(n,p) \to J^k(m,p)$ denote the canonical projection.

Prop.

$\Sigma^i_j \subset J^1(n,p)$ is an RA^1-invariant submfd of codimension

$$j(p-m+j) + i(p-n+i) - j(p-n+i).$$

Furthermore, if $I = (i_1, 0)$ or $J = (j_1, 0)$, then $\Sigma^I_J \subset J^k(n,p)$ is a submfd of codimension

$$\mathrm{cod}\Sigma^{\mathrm{I}} + \mathrm{cod}\Sigma^{\mathrm{J}} - \mathrm{j}_1(\mathrm{p} - \mathrm{n} + \mathrm{i}_1),$$

where codΣ^I and cosΣ^J denote cod.s of $\Sigma^I \subset J^k(n,p)$ and $\Sigma^J \subset J^k(m,p)$ respectively.

Then, we pose some questions.

Q (1) When $\Sigma^I_J = \emptyset$?
(2) In general, is Σ^I_J a submfd of $J^k(n,p)$?
(3) If Σ^I_J is a submfd, then calculate the codimension of $\Sigma^I_J \subset J^k(n,p)$.

! Note that
(1) Assume $m \geq q$. $\Sigma^i_{m-p} = \emptyset$ if $i \neq n-p$. Furthermore,
$\Sigma^{n-p+1,1,1,0}_{m-p+1,1,0} = \emptyset$ and $\Sigma^{n-p+1,1,0}_{m-p+1,1,1,0} = \emptyset$.
(2) $\Sigma^i_j = \emptyset$ if i does not satisfies that
$m - j \leq n - i \leq m - j + \min\{n - m,\ p - m + j\}$.

Our map situation is that $p = q$. Let us apply Relative Transversality theorem for a generic map $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$:

$$
\begin{array}{rl}
j^k f|_{\mathbb{R}^m \times \{0\}} \colon \mathbb{R}^m \times \{0\} \to & J^k(n,p) \\
& \cup \\
& \Sigma^I_J
\end{array}
$$

For a map germ $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$, the above prop. shows that

$$\mathrm{cod}\Sigma^{n-p+1,0}_{m-p+1,0} = n - m + 1.$$

Then, Relative transversality theorem implies that a stable map germ $f\colon (R^n_m, 0) \to (\mathbb{R}^p, 0)$ admits $\Sigma^{n-p+1,0}_{m-p+1,0}$ type singularity $\overset{\text{iff}}{\Leftrightarrow} m \geq n - m + 1 \Leftrightarrow m \geq (n+1)/2$.

Example For a stable map germ $f\colon (R^5_3, 0) \to (\mathbb{R}^3, 0)$ is relative - equivalent to one of the following germs

$$(a,b,x,y_1,y_2) \to \begin{cases} (a,b,x), \ \Sigma_0^{2,0} \ (\mathbf{Regular}, \text{Regular}) \\ (a,b,x^2+y_1+y_2), \ \Sigma_{1,0}^{2,0} \ (\mathbf{Regular}, \text{Fold}) \\ (a,b,x^3+ax+y_1+y_2), \ \Sigma_{1,1,0}^{2,0} \ (\mathbf{Regular}, \text{Cusp}) \\ (a,b,x^4+ax^2+bx+y_1+y_2), \ \Sigma_{1,1,1,0}^{2,0} \ (\mathbf{Regular}, \text{Fold}) \\ (a,b,x^2 \pm y_1^2 \pm y_2^2 + ay_1 + by_2), \ \Sigma_{1,0}^{3,0} \ (\mathbf{Fold}, \text{Fold}) \end{cases}$$

Example For a stable map germ $f\colon (R^4_3, 0) \to (\mathbb{R}^2, 0)$ is relative - equivalent to one of the following germs

$$(a,x_1,x_2,y) \to \begin{cases} (a,x_1), \ \Sigma_{1,0}^{2,0} \ (\mathbf{Regular}, \text{Regular}) \\ (a,x_1^2 \pm x_2^2 + y), \ \Sigma_{2,0}^{2,0} \ (\mathbf{Regular}, \ \text{Fold}) \\ (a,x_1^3 \pm ax_1 \pm x_2^2 + y), \ \Sigma_{2,1,0}^{2,0} \ (\mathbf{Regular}, \ \text{Cusp}) \\ (a,x_1^2 \pm x_2^2 \pm y^2 + ay), \ \Sigma_{2,0}^{3,0} \ (\mathbf{Fold}, \ \text{Fold}) \end{cases}$$

Example For a stable map germ $f\colon (R^3_2, 0) \to (\mathbb{R}^3, 0)$ is relative - equivalent to one of the following germs

$$(a,x,y) \to \begin{cases} (a,x,y), \ \Sigma^0_0 \ (\textbf{Regular}, \text{Regular}) \\ (a,x^2,ax+y), \ \Sigma^{1,0}_{1,0} \ (\textbf{Fold}, \ \text{Whitney Umbrella}) \end{cases}$$

Example For a stable map germ $f\colon (R^4_2, 0) \to (\mathbb{R}^3, 0)$ is relative - equivalent to one of the following germs

$$(a,x,y_1,y_2) \to \begin{cases} (a,x,y_1), \ \Sigma^{1,0}_0 \ (\textbf{Regular}, \text{Regular}) \\ (a,x^2+y_1,ax+y_2), \ \Sigma^{1,0}_{1,0} \ (\textbf{Regular}, \ \text{Whitney Umbrella}) \end{cases}$$

Cf. A map germ $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ $\mathcal{A}$-eq. to $f = (a, x^2, ax)$ is called Whitney Umbrella (or Cross Cap).