C^1-triangulation of semialgebraic sets and de Rham homotopy theory

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Content

- Semialgebraic sets and maps: C^1-Triangulation
- Idea of the proof
- De Rham homotopy theory of semialgebraic sets (Kontsevich-Soibelman)

This is a joint work with Masahiro Shiota (Nagoya). 

Semialgebraic sets and maps

Def.

A **semialgebraic set** is a subset X of some $\mathbb{R}^m$ defined by finitely many polynomial equalities and inequalities:

$$X = \{\ x \in \mathbb{R}^m \mid f_i(x) = 0,\ g_j(x) > 0\ \}$$

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- Closed under Boolean operations.
- Closed under linear projection $p : \mathbb{R}^m \to \mathbb{R}^n$
  (X semialg. $\Longrightarrow p(X)$ semilalg.)

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Def.

A **semialgebraic map** $f : X \to Y$, where $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, is a map such that the graph of f is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$.

Semialgebraic sets and maps

Def.

A **semialgebraic C^k map** $X \to Y$ is the restriction of a semialg. C^k map

$$U \to \mathbb{R}^n \quad (U \text{ is a semialg. open nbd of } X \subset \mathbb{R}^m).$$

- $U = \mathbb{R}^m$ by using *semialgebraic C^k bump functions*

Semialgebraic sets and maps

(Convention)
X or f is *locally semialgebraic* if it's semialgebraic over any balls $B_r(p) = \{x \in \mathbb{R}^m,\ |x - p| \leq r\}$. As a convention in this talk, we use *semialgebraic* to mean *locally semialgebraic*.

(Later, we may work on more wider class of sets, *piecewise algebraic spaces*, introduced by Kontsevich).

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Also possible to work in

- subanalytic sets and maps (Hironaka)
- $\mathfrak{X}$-category (Shiota)
- o-minimal category over any real closed field

Semialgebraic sets and maps

A **semialgebraic triangulation** of a locally closed semialgebraic set X is the pair (K, f) of

- K: a locally finite simplicial complex with $|K| \subset \mathbb{R}^m$ for some m
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- The restriction on any open simplex

$$f|_{\mathrm{Int}\,\sigma} : \mathrm{Int}\,\sigma \to f(\mathrm{Int}\,\sigma)$$

is semialg. diffeomorphism.

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- For each simplex $\sigma \in K$, the restriction

$$f|_\sigma : \sigma \to f(\sigma)$$

is of C^1, but not diffeo. on some semialg. subsets of less dim.

Semialgebraic sets and maps

Theorem 3 (Ohmoto-Shiota)

Let X be a locally closed semialg. set with a semialg. triangulation (K, f), and $\varphi : X \to Y$ a semialg. C^0 map. Then $\exists\, \chi$ semialg. homeo s.t. χ preserves $\sigma \in K$ and the composed map

$$|K| \xrightarrow{\chi\ \simeq} |K| \xrightarrow{f\ \simeq} X \xrightarrow{\varphi} Y \qquad \text{is of class } C^1.$$

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- If $Y = X$ and $\varphi = id_X$, Thm 2 follows from Thm 3;
- If we apply Thm 3 to

$$id \times f \times (\varphi \circ f) : |K| \to |K| \times X \times Y,$$

then χ, $f \circ \chi$ and $\varphi \circ f \circ \chi$ are of C^1 simultaneously.

Semialgebraic sets and maps

Panel Beating = 鈑金

Semialgebraic sets and maps: Application

M: semialgebraic manifold
$X \subset M$: cpt semialgebraic set of $\dim = q$ with a triangulation (K, f)
$\omega \in A^q(M)$: differential q-form on M

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The integral of ω over X is defined as the *improper* integral

$$\int_X \omega := \sum_{\dim \sigma = q} \lim_{\epsilon \to 0} \int_{\mathrm{Int}\ _\epsilon \sigma} f^* \omega$$

but the convergence is not obvious.

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Use the Lebesgue converg. thm and the fact that X has a finite volume, that is verified by the Łojasiewicz inequality
$\Rightarrow$ **the theory of semialgebraic currents** (Federer-Hardt)

$$\int_X : \Omega^q(M) \to \mathbb{R}.$$

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The integral of ω over X is defined as *definite* integral:

$$\int_X \omega := \sum_{\dim \sigma = q} \int_{|\sigma|} (f \circ \chi)^* \omega$$

(that answers to a question posed by Tatsuo Suwa).

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- X, ω defned over $\mathbb{Q} \Rightarrow \int_X \omega \in \mathrm{Period} \subset \mathbb{R}$.

Proof

Given a semialg. C^0-map $\varphi : X \to Y$ and a triangulation (K, f_1) of X. Put $f = \varphi \circ f_1 : |K| \to Y$ and extend it to

$$|K| \subset \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \supset Y$$

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We construct a panel beating $\chi : \mathbb{R}^m \to \mathbb{R}^m$ preservng $\forall \sigma \in K$ s.t.

$$f \circ \chi : \mathbb{R}^m \longrightarrow \mathbb{R}^n \text{ is of class } C^1$$

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$$|K| \subset \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \supset Y$$ **semialg. C^0 map**

Let $\overline{V}$ (V:smooth) be the smallest semialgebraic set ('**bad subset**') s.t.

$f : \mathbb{R}^m - \overline{V} \longrightarrow \mathbb{R}^n$ **is of class C^1 and $f|_V$ is of C^1.**

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- By stratification theory, we see $d = \dim V < m$.
- We find a panel beating χ around V so that it preserves $\forall \sigma \in K$ and

$$f \circ \chi : \mathbb{R}^m - S \longrightarrow \mathbb{R}^n \text{ is of class } C^1$$

where $S \subset \overline{V}$ is some semialg. subset of dim $< d$.

Proof

Panel beating χ around V so that $f \circ \chi : \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1 ??

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Example. $f(x) = \sqrt{|x|}$ is not C^1 at $x = 0$.

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Example. $f(x) = \sqrt{|x|}$ is not C^1 at $x = 0$.

$$\chi(x) = \begin{cases} x^4 & (x \geq 0) \\ -x^4 & (x \leq 0) \end{cases} \text{ semialg. homeo } \implies f \circ \chi(x) = x^2.$$

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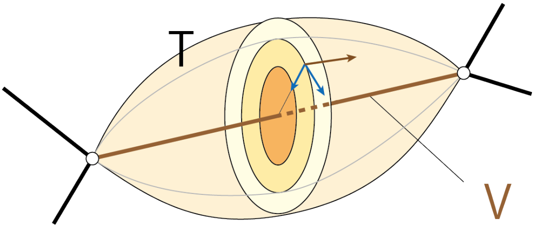
Panel beating χ around V so that $f \circ \chi : \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1 ??

Take a C^1-tube $(T, \pi : T \to V, \rho : T \to \mathbb{R}_{\geq 0})$ along V.



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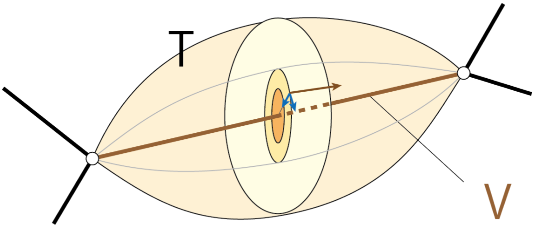
Take a C^1-tube $(T, \pi : T \to V, \rho : T \to \mathbb{R}_{\geq 0})$ along V.
Shrink the tube via a semialg. homeo. $\chi : T \to T$
using the cone structure of T.

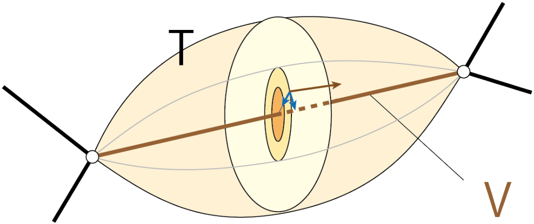


Proof

$\boldsymbol{x} = (x_1, \ldots, x_d)$: coord. of V, $\boldsymbol{u} = (u_1, \cdots, u_{m-d})$: the fibre

$\frac{\partial}{\partial u_k} f \circ \chi(\boldsymbol{x}, \boldsymbol{u}) \to 0$. OK, $\frac{\partial}{\partial x_j} f \circ \chi(\boldsymbol{x}, \boldsymbol{u}) \to \frac{\partial}{\partial x_j} f|_V(\boldsymbol{x})$?? $(\boldsymbol{u} \to 0)$

$\Longrightarrow$ $f \circ \chi$ is of C^1 along V; $d(f \circ \chi)(\boldsymbol{x}, \boldsymbol{u}) \to df|_V(\boldsymbol{x})$. ???



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Set

$$S := \{\boldsymbol{x} \in V |\ \frac{\partial f \circ \chi}{\partial x}(\boldsymbol{x}, \boldsymbol{u}) \not\to \frac{\partial f|_V}{\partial x}(\boldsymbol{x}) \text{ as } \boldsymbol{u} \to 0\}$$

Proof

- $\frac{\partial}{\partial x_j} f \circ \chi(\boldsymbol{x}, \boldsymbol{u}) \to \frac{\partial}{\partial x_j} f|_V(\boldsymbol{x})$??

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- S is semialgebraic
- it suffices to show $\dim S < d$
- Suppose $\dim S = d$, then deduce the contradiction.
- The main tool is the curve selection lemma.

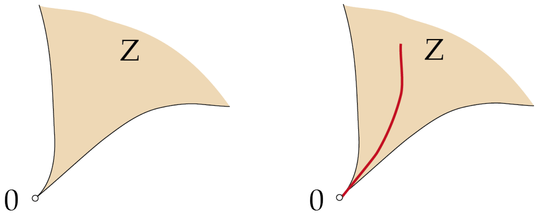
Proof

Curve Selection Lemma

$Z \subset \mathbb{R}^m$: semialgebraic set s.t. $0 \in \overline{Z}$

$\Longrightarrow$

$\exists\, \gamma : (0, \epsilon) \to Z$: semialgebraic C^k-curve s.t. $\lim_{t \to 0} \gamma(t) = 0$



Proof

Suppose $Z = \{(\boldsymbol{x}, \boldsymbol{u}) \in T | \ \frac{\partial f_1 \circ \chi}{\partial x_1}(\boldsymbol{x}, \boldsymbol{u}) - \frac{\partial f_1|_V}{\partial x_1}(\boldsymbol{x}) > \epsilon_1\} \neq \emptyset$
semilag. set s.t. $S \times \{0\} \subset \overline{Z}$

Proof

Suppose $Z = \{(\boldsymbol{x}, \boldsymbol{u}) \in T |\ \frac{\partial f_1 \circ \chi}{\partial x_1}(\boldsymbol{x}, \boldsymbol{u}) - \frac{\partial f_1|_V}{\partial x_1}(\boldsymbol{x}) > \epsilon_1\} \neq \emptyset$
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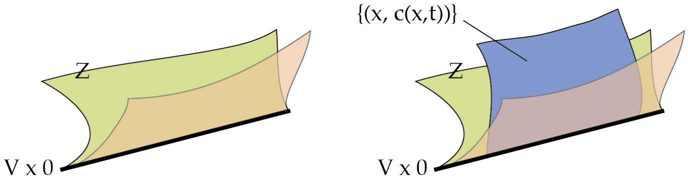
Curve selection lemma (family version)
$\exists\, U \subset V$: semialg. open set,
$\exists\, \gamma : U \times (0, \epsilon) \to Z$: semialg. C^1-map s.t. $\lim_{t \to 0} \gamma(\boldsymbol{x}, t) = \boldsymbol{x}$.



Proof

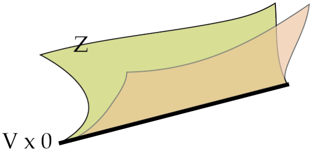
$$\left|\frac{\partial f_1\circ\chi}{\partial x_1}(\gamma(\boldsymbol{x},t)) - \frac{\partial f_1|_V}{\partial x_1}(\boldsymbol{x})\right| > \epsilon_1$$

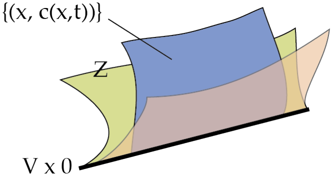
$$f_1(\gamma(\boldsymbol{x},t) - f_1(\gamma(\boldsymbol{a},t)) = \int \frac{\partial f_1\circ\chi}{\partial x_1}(\gamma(\boldsymbol{x},t))\,dx_1$$

$$-)\qquad f_1(\boldsymbol{x},0) - f_1(\boldsymbol{a},0) = \int \frac{\partial f_1|_V}{\partial x_1}(\boldsymbol{x})\,dx_1$$

tends to 0 uniformly positive

that makes a contradiction.





Piecewise algebraic space

Any semialg. sets are assumed to be a subset in some $\mathbb{R}^m$.
However our proof depends only on the local problems. We can work in a bit more wider class of sets:

Def. (Kontsevich-Soibelman)

An **piecewise algebraic space** is a locally compact Hausdroff space $X = \cup_\alpha X_\alpha$ obtained by gluing countably many semialgebraic sets X_α in $\mathbb{R}^{m_\alpha}$ via semialgebraic homeomorphisms on their constructible open subsets:

$$\psi_\alpha^\beta : X_\alpha \supset X_{\alpha\beta} \overset{\simeq}{\to} X_{\beta\alpha} \subset X_\beta$$

A PA space may not be in some $\mathbb{R}^m$. A compact PA space is a semialg. sets in $\mathbb{R}^m$.

Piecewise algebraic differential forms

AIM: build the de Rham theory for semialgebraic set (or PA space) X

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We take the **pair (f, dt) of function and differential form**.

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How to define df of a semialg. C^0-function $f : X \to \mathbb{R}$?

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Def.

A **minimal semialg. diff. form** on a compact semialg. set X is a formal finite sum $\sum h_i \cdot \omega_i^{\min}$ where $h_i : X \to \mathbb{R}$ is a semialg. C^0 function and

$$\omega^{\min} := (f, \omega) \quad (f : X \to M\text{: semialg. } C^0 \text{ map, } \omega \in A^q(M))$$

with M a semialgebaric manifold (can be $\mathbb{R}^n$). For a PA space X, glue such forms via a natural equiv. on overlaps.

Piecewise algebraic differential forms

We have a CDGA (commutative diff. graded algebra)

$$\Omega^*_{\min}(X) := \{ \sum h_i \cdot (f_i, \omega_i) \ \text{minimal forms} \} / \sim$$

($\sim$ is explained later).

Product:

$$\omega_1^{\min} \wedge \omega_2^{\min} := (f \times g : X \to M_1 \times M_2, \ \ p_1^* \omega_1 \wedge p_2^* \omega_2).$$

Differential:

$$d(h \cdot \omega^{\min}) := dh \wedge \omega^{\min} + h \cdot (f, d\omega).$$

Pullback via $\varphi : X' \to X$

$$\varphi^* \omega^{\min} = (f \circ \varphi, \omega) \ \in \Omega^*_{\min}(X').$$

Piecewise algebraic differential forms

Given $\omega^{\min} = (f, \omega) \in \Omega^q_{\min}(X)$ and a semialg. C^0 map $\sigma : \Delta^q \to X$ (semialg. singular simplex), take a panel beating χ so that

$$f \circ \sigma \circ \chi : \Delta^q \to \Delta^q \to X \to M \text{ is of } C^1.$$

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$\omega_1^{\min} \sim \omega_2^{\min}$ if $\int_\sigma \omega_1^{\min} = \int_\sigma \omega_2^{\min}$ for any q-simplex $\sigma \in S_q^{sa}(X)$.

Piecewise algebraic differential forms

- $\Omega^*_{\min}(X)$ does not satisfy the Poincaré lemma
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Ex. $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto t = xy$. $f_*(xdy) = t \log t$.

Piecewise algebraic differential forms

Def. (cf. Kontsevich-Soibelman (2009))

A **picewise algebraic diff. q-form (PA form) on X** is a pair $\tau^{PA} = (\varphi, \omega)$ of

- $\varphi : Y \to X$: a semilag. proper C^0 map with $\dim Y - \dim X = k$;
- $\omega^{\min}$: a minimal $(q+k)$-form on Y.

Denote by $\Omega^*_{PA}(X)$ the CDGA of PA forms on X up to the same equiv. $\sim$.

Piecewise algebraic differential forms

Def. (cf. Kontsevich-Soibelman (2009))

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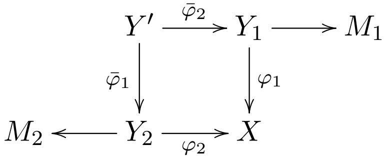
product: for $\tau_j^{PA} = (Y_j \to X, \omega_j^{\min})$ $(j = 1, 2)$,

$$\tau_1^{PA} \wedge \tau_2^{PA} := (\varphi : Y' \to X,\ \bar\varphi_1^* \omega_1^{\min} \wedge \bar\varphi_2^* \omega_2^{\min}),$$



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Integral of a PA q-form $\omega^{PA} = (\varphi, \omega^{\min})$ over a semialg. singular simplex $\sigma : \Delta^q \to X$ is defined by

$$\int_\sigma \omega^{PA} := \int_{\varphi^{-1}(\sigma(\Delta^q))} \omega^{\min}.$$

These are subgroups of the semialg. singular cochain group:

$$\Omega^q_{\min}(X) \subset \Omega^q_{PA}(X) \subset S^q_{sa}(X).$$

- $\Omega^*_{PA}(X)$ does satisfy the Poincaré lemma.

Piecewise algebraic differential forms

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Remark

- The original definition of PA forms (due to KS) is much involved.
- For compact semilagebraic sets X, there has rigorously been established a framework of PA forms and their integrations using **semialgebraic currents** by
  Hardt, Lambrechts, Turchin, Volić, Geom & Top. (2011) 70pp.

Sullivan functor ($R = \mathbb{Q}$):
for a simplicial complex $X = |K| \subset \mathbb{R}^m$

$$A_{PL}(X, R) = \bigoplus_k \{ \{\omega_\sigma\}_{\sigma \in K},\ \omega_\sigma : \text{polynomial } k\text{-form on } \sigma\}$$

- $\omega_\sigma = \sum p_I(t_0, \cdots, t_m) dt_{i_1} \wedge \cdots dt_{i_k}$,

where p_I are polynomials on a simplex σ with coefficients in R.
- $\omega_\tau = (\omega_\sigma)|_\tau$ for $\tau \prec \sigma$.

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where p_I are polynomials on a simplex σ with coefficients in R.
- $\omega_\tau = (\omega_\sigma)|_\tau$ for $\tau \prec \sigma$.

- $A_{PL}(X, \mathbb{Q})$ carries the information about $\pi_*(X) \otimes \mathbb{Q}$.
- For a smooth manifold X, $A_{PL}(X, \mathbb{R})$ is quasi-isomorphic to the de Rham complex $\Omega^*(X)$

Real homotopy theory for PA spaces

Theorem 4 (cf. Kontsevich-Soibelman (2009), Hardt etc (2011))

Let X be a PA space. Then $\Omega^*_{PA}(X)$ is weak-equivalent to $A_{PL}(X, \mathbb{R})$.

- Theorem has rigorously been proven for compact X by Hardt etc.
- Theorem was used for proving the formality of little cube operads (Kontevich).
- Our approach using panel beating brings a significant simplification of the proof; it works even for non-compact case.