Constructions of contact structures via the moment map I

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Contact submanifolds

(M, ξ), (N, η): contact manifolds.

Definition

An embedding $f : M \to N$ is said to be a contact embedding if $TM \cap \eta|_M = \xi$. Then (M, ξ) is called a contact submanifold of (N, η).

Let $\xi = \ker \alpha$ and $\eta = \ker \beta$.

- $TM \cap \eta|_M = \xi \Leftrightarrow \ker(f^* \beta) = \ker \alpha$.

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- Since $TV = JTV$, $TK \cap \xi_0|_K = TK \cap JTK$.
- $(K, TK \cap \xi_0|_K)$ is a contact submanifold.

Examples 2-2

Brieskorn singularity (Milnor) $z_1^p + z_2^q + z_3^r = 0.$

1. Spherical $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1,$
   $(2, 2, r)$: $A_{r-1} \Rightarrow$ tight str on $L(r, r-1),$
   $(2, 3, 5)$: $E_8 \Rightarrow$ tight str on $\Sigma(2, 3, 5).$
2. Euclidean $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \Leftrightarrow$ Simple elliptic sing
3. Hyperbolic $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$
   $(4, 4, 4)$: S^1-bundle over Σ_3 with the Euler class $-4.$

Examples 2-3

Simple elliptic singularity (Saito, Neumann, cf. Mori)

$\tilde{E}_6 : z_1^3 + z_2^3 + z_3^3 + \lambda_1 z_1 z_2 z_3 = 0 \ ({\lambda_1}^3 + 27 \neq 0)$,
$\tilde{E}_7 : z_1^2 + z_2^4 + z_3^4 + \lambda_2 z_1 z_2 z_3 = 0 \ ({\lambda_2}^4 - 64 \neq 0)$,
$\tilde{E}_8 : z_1^2 + z_2^3 + z_3^6 + \lambda_3 z_1 z_2 z_3 = 0 \ ({\lambda_3}^6 - 432 \neq 0)$.

Cusp singularity (Laufer, Hirzebruch, Neumann, K, cf. Mori)

$T_{pqr} : z_1^p + z_2^q + z_3^r + \lambda z_1 z_2 z_3 = 0 (\lambda \neq 0, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1)$.

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- $(T_A, \ker(dy + mzdx))$, $A = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$, $m = 3, 2, 1$.

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- $(T_A, \ker(\beta_+ + \beta_-))$, Anosov contact str,
  $A = \begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix}$.

Examples 3

Via moment polytope of S^5 (Mori, K, Furukawa)
The followings can be contact submfds of (S^5, ξ_0).

1. An overtwisted contact str on S^3,
2. Tight contact strs on T^3,
3. Some tight contact strs on T^2 bundles over S^1.

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- **The construction of the last ones can also explain the links of simple elliptic and cusp singularities.**

Main Theorem

Theorem

The singularity link of $f_{m,k}(z) = 0$ *is contactomorphic to* $(T_{A_{m,k}}, \ker(\beta_+ + \beta_-))$ *if* $k \neq 0$, *and to* $(T_{A_{m,0}}, \ker(dy + mzdx))$ *if* $k = 0$, *where*

$$\begin{aligned} f_{1,(k_1)}(z) &= z_1^2 + z_2^3 + z_3^{6+k_1} + z_1z_2z_3 \\ f_{2,(k_1,k_2)}(z) &= z_1^2 + z_2^{4+k_1} + z_3^{4+k_2} + z_1z_2z_3 \\ f_{3,(k_1,k_2,k_3)}(z) &= z_1^{3+k_1} + z_2^{3+k_2} + z_3^{3+k_3} + z_1z_2z_3. \end{aligned}$$

Main Theorem 2

$$A_{m,k} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_m \\ 0 & 1 \end{pmatrix},$$

where $m \in \mathbb{Z}_{>0}$, $k = (k_1, \cdots, k_m) \in (\mathbb{Z}_{\geq 0})^m$.

- $A_{m,k}$ is a hyperbolic matrix if $k \neq 0$,
- $A_{m,0}$ is a parabolic matrix.

N. Kasuya, The canonical contact structure on the link of a cusp singularity, Tokyo J. Math. Vol. 37, No. 1, 2014, 1-20.

The moment polytope

$(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3)$: the polar coordinates.
$S^5 = \{r_1^2 + r_2^2 + r_3^2 = 1\} \subset \mathbb{C}^3$.

$$\phi : S^5 \to \mathbb{R}^3; (z_1, z_2, z_3) \mapsto (r_1^2, r_2^2, r_3^2)$$

is called the moment map. $\Delta := \phi(S^5)$ is the moment polytope.
The projection ϕ is T^3-fibration over $\mathrm{Int}\Delta$, T^2-fibration over edges, S^1-fibration on vertices.

The moment polytope 2

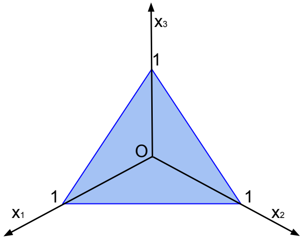

Figure: Moment polytope

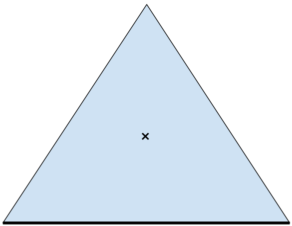

Figure: standard 3-sphere

The principle

Let $c : [0,1] \to \mathrm{Int}\Delta$ be a curve.
$f : T^2 \times [0,1] \to S^5$: an embedding defined by
"the slice section" $\{p\theta_1 + q\theta_2 + r\theta_3 = 0\}$ over c.

Lemma

$(T^2 \times [0,1], \ker(f^*\alpha_0))$ *is positive contact iff the curve* c *rotates (counter) clockwise around the point* $\frac{1}{p+q+r}(p,q,r)$, *when it is inside (outside)* Δ.

Mori's example

A. Mori, The Reeb foliation arises as a family of Legendrian submanifolds at the end of a deformation of the standard S^3 in S^5, C. R. Acad. Sci. Paris, 350 (2012) 67-70.

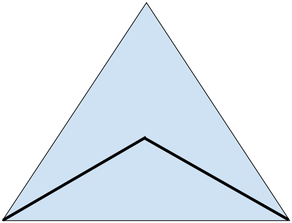

Figure: Reeb foliation

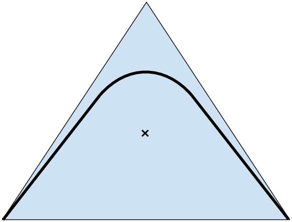

Figure: OT 3-sphere

The 3-torus case

(T^3, η_n), $\beta_n = \sin(2n\pi z)dx + \cos(2n\pi z)dy$.
We embed (T^3, η_1) in (S^5, ξ_0) as a contact submfd.
Define an embedding by

$$f : T^3 \to S^5; (x, y, z) \mapsto (r_1, r_2, r_3, x, y, -x-y),$$

$$3\begin{pmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \varepsilon \sin(2\pi z) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \varepsilon \cos(2\pi z) \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

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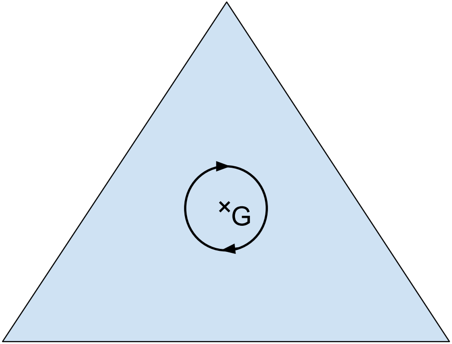
- $f^*({r_1}^2 d\theta_1 + {r_2}^2 d\theta_2 + {r_3}^2 d\theta_3) = \varepsilon\beta_1$.
- $(T^3, \ker f^*\alpha_0) \cong (T^3, \eta_1 = \ker \beta_1)$.

The 3-torus case 2

Let us project $f(T^3)$ by ϕ on Δ.
$G = (1/3, 1/3, 1/3)$: the barycenter of Δ.



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- The contact condition is equivalent to the negativity of the angular momentum of the curve l: $c(z)$ $(z \in [0, 1])$, around G.

The A_{k-1}-singularity link

The link of $z_1 z_2 - z_3^k = 0$ is $(L(k, k-1), \xi)$.
Furukawa showed this by the following argument.
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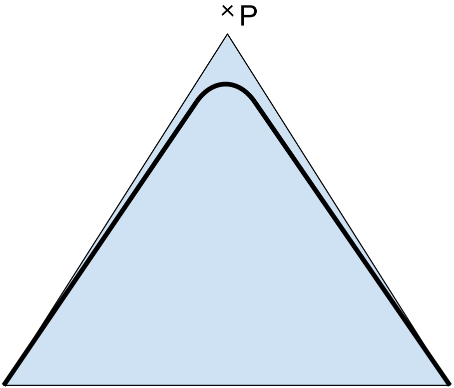
- Slice sections over $c([0, \frac{1}{2}])$ and $c([\frac{1}{2}, 1])$ are the standard contact solid tori pasted by the linear map $\begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix}$. $\Rightarrow (L(k, k-1), \xi)$.

The A_{k-1}-singularity link 2



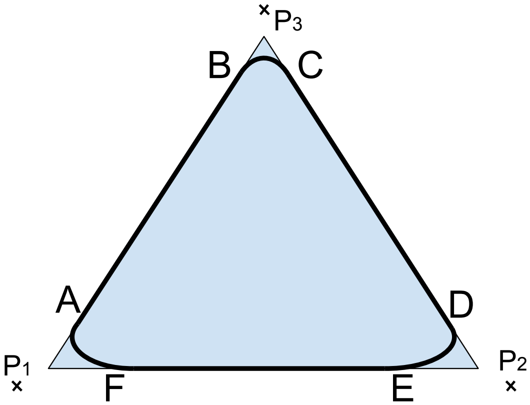
Some T^2 bundles over S^1

Let $A = \begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix}$.
Furukawa embedded $(T_A, \ker(\beta_+ + \beta_-))$ in (S^5, ξ_0).
The slice section over the curve $c : [0,1] \to \Delta$ is defined by

$$\begin{cases} \{(r-1)\theta_3 - \theta_1 - \theta_2 = 0\} \, (t \in (\frac{1}{6}, \frac{1}{3})), \\ \{(q-1)\theta_2 - \theta_3 - \theta_1 = 0\} \, (t \in (\frac{1}{2}, \frac{2}{3})), \\ \{(p-1)\theta_1 - \theta_2 - \theta_3 = 0\} \, (t \in (\frac{5}{6}, 1)). \end{cases}$$

This is also valid for the case $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

Some T^2 bundles over S^1 2



The rough idea

Let $L_\lambda = S^5 \cap \{z_1^p + z_2^q + z_3^r - \lambda z_1 z_2 z_3 = 0\}$.
When $\lambda \to \infty$, $\phi(L_\lambda)$ approaches to $\partial\Delta$.

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- Namely, we obtained Furukawa's model.

$$L_\lambda \cong \begin{cases} (T_A, \ker(\beta_+ + \beta_-)) & (\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1) \\ (T_A, \ker(dy + mzdx)) & (\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1). \end{cases}$$

The perturbation

To prove Main Theorem, we construct an isotopy of contatct submanifolds between L_λ and Furukawa's model for large λ. Then, by Gray stability, they are contactomorphic. Let $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a bump function supported on $\{s \in \mathbb{R} \mid 1 - 2\delta \leq s\}$ and $\varphi \equiv 1$ on $\{s \in \mathbb{R} \mid 1 - \delta \leq s\}$ with $0 < \delta < \frac{1}{5}$.

$F_\lambda := z_1 z_2 z_3 - \dfrac{1}{\lambda}(z_1^p + z_2^q + z_3^r)$ and

$G_\lambda := z_1 z_2 z_3 - \dfrac{1}{\lambda}(\varphi(r_1^2) z_1^p + \varphi(r_2^2) z_2^q + \varphi(r_3^2) z_3^r)$.

The perturbation 2

$H_t := (1-t)F_\lambda + tG_\lambda$.
For large λ, $H_t^{-1}(0)$ is a contact submanifold.

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2. On $\left\{|z_1|, |z_2|, |z_3| < \sqrt{1-\frac{1}{2}\delta}\right\}$, $H_t^{-1}(0)$ is close to $\{z_1 z_2 z_3 = 0\}$ in the sense of C^∞ topology, and the contactness is an open condition.

This completes the proof.

Remark

I. Zharkov used a similar argument to construct Lagrangian torus fibrations of Calabi-Yau hypersurfaces. The figure below represents the elliptic curve $x^3 + y^3 + z^3 + txyz = 0$ in $\mathbb{C}P^2$.
I. Zharkov, Torus fibrations of Calabi-Yau hypersurfaces in toric varieties, Duke Math. J. Vol. 101, No. 2, 2000, 237-257.

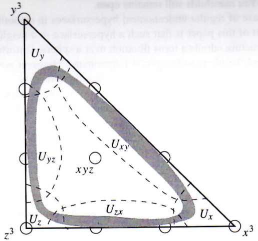

FIGURE 1. The image of the elliptic curve $txyz + x^3 + y^3 + z^3 = 0$ (shaded area) in $\mathbb{CP}^2$ under the moment map

Thank you for your attention!