Definition and Self-Adjointness of the Stochastic Airy Operator *

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Abstract

In this note, it is shown that the stochastic Airy operator, which is the "Schrödinger operator"on the half-line whose potential term consists of Gaussian white noise plus a linear term tending to $+\infty$, can naturally be defined as a generalized Sturm-Liouville operator and that it is self-adjoint and has purely discrete spectrum with probability one. Thus "stochastic Airy spectrum"of Ramírez, Rider and Virág is the spectrum of an operator in the ordinary sense of the word.

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1 Introduction.

In [2], Dumitriu and Edelman considered the following random matrix

$$H_n^\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix} g_1 & \chi_{(n-1)\beta} & 0 & \cdots & & \cdots & 0 \\ \chi_{(n-1)\beta} & g_2 & \chi_{(n-2)\beta} & 0 & \cdots & & 0 \\ & \cdots & & \cdots & \cdots & & \cdots \\ & \cdots & & \cdots & \cdots & & \cdots \\ 0 & \cdots & & \cdots & \chi_{2\beta} & g_{n-1} & \chi_\beta \\ 0 & \cdots & & \cdots & 0 & \chi_\beta & g_n \end{bmatrix}, \tag{1}$$

where $\beta > 0$ is a real number and the random variables $g_1, \dots, g_n$ obey the normal distribution $N(0,2)$, whereas $\chi_{j\beta}$ $(j = 1, \dots, n-1)$ obeys the χ-distribution with parameter $j\beta$, all random variables being independent each other. Here by definition, a random variable X obeys the χ-distribution if and only if X^2 obeys the χ^2- distribution. They called H_n^β the β-ensemble, and proved that the n eigenvalues of H_n^β have the joint probability density of the form

$$P_n^\beta(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_n^\beta} e^{-\frac{\beta}{4}\sum_{k=1}^n \lambda_k^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta, \tag{2}$$

Z_n^β being the normalizing constant. For the special values $\beta = 1, 2, 4$, (2) represents the joint probability density of the eigenvalues of the random matrix GOE, GUE and GSE respectively (see e.g. [7]). It was found out by Trotter [16] that these three well studied matrix ensembles can be transformed into the tridiagonal form (1), and in this form, the general β-ensemble interpolate and extrapolate those three ensembles.

Concerning the limiting distribution of the largest k eigenvalues $\lambda_1^{(n)} \geq \cdots \geq \lambda_k^{(n)}$ of H_n^β, Ramírez, Rider and Virág [11] proved that as $n \to \infty$, the joint distribution of $\{n^{1/6}(2\sqrt{n} - \lambda_j^{(n)})\}_{j=1}^k$ converges to that of the smallest k eigenvalues $\Lambda_0 \leq \cdots \leq \Lambda_{k-1}$ of the "Schrödinger operator"

$$H = -\frac{d^2}{dt^2} + t + \frac{2}{\sqrt{\beta}} B'_\omega(t) \;, \quad t \geq 0 \tag{3}$$

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with Dirichlet boundary condition at $t = 0$, where $\{B_\omega(t)\}_{t\geq 0}$ is the standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $B'_\omega(t)$ is the formal derivative of its sample function.

Since sample functions of the Brownian motion are not differentiable, the expression (3) itself needs a justification. Ramírez et. al. consider H to be a random linear mapping from the function space

$$H^1_{loc}(\mathbf{R}_+) := \{f;\ f(\cdot) \text{ is absolutely continuous on } [0,\infty) \text{ and } f' \in L^2_{loc}(\mathbf{R}_+)\}$$

into the space of Schwartz distribution. They further define the eigenvalues and eigenfunctions of H to be pairs $(\lambda, f) \in \mathbf{R} \times L^*$ such that $Hf = \lambda f$ holds, both sides of which being interpreted as Schwartz distributions. Here the function space L^* is defined by

$$\begin{aligned} L^* = \{f \quad ; \quad & f(\cdot) \text{ is absolutely continuous on } [0,\infty),\ f(0)=0, \text{ and} \\ & \int_0^\infty \{|f'(t)|^2 + (1+t)|f(t)|^2\}dt < \infty\}\ . \end{aligned} \tag{4}$$

Actually, Schwartz distribution is irrelevant in formulating the eigenvalue problem for H, and it turns out that H is realized quite naturally as a self-adjoint operator in the Hilbert space $L^2(\mathbf{R}_+)$. Namely we have the following

Theorem 1 *For each ω, H can be realized as a closed symmetric operator in $L^2(\mathbf{R}_+)$, which is self-adjoint with probability one and has purely discrete spectrum.*

Thus what Ramírez et.al. called "stochastic Airy spectrum"is the spectrum of a self-adjoint operator in the ordinary sense of the word. We shall prove Theorem 1 as an immediate corollary of the following more general result.

Theorem 2 *Let $p(t) \geq 0$ be a real valued continuous function on $\mathbf{R}_+$ such that $\liminf_{t\to\infty} p(t)/t^\alpha > 0$ for some $\alpha > 0$. Further let $\{X_\omega(t)\}$ be a fractional Brownian motion of Hurst parameter $h \in (0,1)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then under the Dirichlet boundary condition at $t = 0$,*

$$H := -\frac{d^2}{dt^2} + p(t) + cX'_\omega(t)\ , \quad t \geq 0 \tag{5}$$

can be realized as a closed symmetric operator in the Hilbert space $L^2(\mathbf{R}_+)$, which is self-adjoint and has purely discrete spectrum with probability one.

Here a fractional Brownian motion $\{X_\omega(t)\}$ of Hurst parameter $h \in (0,1)$ is a stochastic process satisfying the following conditions:

(i) $X_\omega(0) = 0$ and $X_\omega(t)$ is continuous with respect to $t \geq 0$ for all $\omega \in \Omega$;

(ii) $\{X_\omega(t)\}$ is a centered Gaussian process;

(iii) $\mathbf{E}[X(t)X(s)] = \frac{1}{2}(t^{2h} + s^{2h} - |t-s|^{2h})$.

See Nourdin [10] for details.

Since the standard Brownian motion is the fractional Brownian motion of Hurst parameter $h = 1/2$, Theorem 1 follows from Theorem 2 by letting $p(t) = t$ and $c = 2/\sqrt{\beta}$.

In the following section, we give a definition of the operator H as a closed symmetric operator. The argument there is purely deterministic. In §3, we consider a random quadratic form

$$\begin{aligned} \mathcal{E}_\omega(u,v) &= \int_0^\infty \{u'(t)\overline{v'(t)} + (p(t) + ca'_\omega(t))u(t)\overline{v(t)}\}dt \\ &\quad - \int_0^\infty c(X_\omega(t) - a_\omega(t))(u(t)\overline{v(t)})'dt\ , \end{aligned} \tag{6}$$

which is defined for $u, v \in L$, where

$$L := \{f \quad ; \quad f(\cdot) \text{ is absolutely continuous on } [0,\infty),\ f(0)=0, \text{ and } \int_0^\infty \{|f'(t)|^2 + (1+p(t))|f(t)|^2\}dt < \infty\} \ . \tag{7}$$

Here we have set, as in [11],

$$a_\omega(t) = \int_t^{t+1} X_\omega(s)ds \ . \tag{8}$$

It will then be shown that for $\mathbf{P}$-almost all ω, $\mathcal{E}_\omega$ is closed, lower semi-bounded and completely continuous. Hence by general theory, there corresponds a self-adjoint operator A_ω which is lower semi-bounded and has purely discrete spectrum. To complete the proof of Theorem2, we then verify that this A_ω is a self-adjoint extension of the symmetric operator defined in §2 and that this symmetric operator is of limit point type at $+\infty$.

2 Definition of H as a symmetric operator.

Let $p(t)$ and $Q(t)$ be real valued continuous functions on $[0,\infty)$ with $Q(0)=0$, and consider the expression

$$H = H(p,Q) = -\frac{d^2}{dt^2} + p(t) + Q'(t) \ , \tag{9}$$

where $Q'(t)$ is the formal derivative of $Q(t)$.

Definition 1 ([8]) *A complex valued function $u(t)$ belongs to the space $\mathcal{C} = \mathcal{C}(p,Q)$ if and only if u is absolutely continuous on $[0,\infty)$ and if there are $\alpha \in \mathbf{C}$ and $v(\cdot) \in L^1_{loc}(\mathbf{R}_+)$ such that*

$$u'(t) = \alpha + Q(t)u(t) + \int_0^t \{p(y)u(y) - Q(y)u'(y) - v(y)\}dy \ . \tag{10}$$

It is clear that v is uniquely determined from u as an element of $L^1_{loc}(\mathbf{R}_+)$. We define $Hu = H(p,Q)u = v$ for each $u \in \mathcal{C}$.

If we define, following Savchuk and Shkalikov [14], the "quasi-derivative"of u by $u^{[1]}(t) = u'(t) - Q(t)u(t)$, then it is easily seen that $u \in \mathcal{C}$ is equivalent to saying that $u(t)$ and $u^{[1]}(t)$ are absolutely continuous and satisfy

$$v(t) := -(u^{[1]}(t))' + (p(t) - Q^2(t))u(t) - Q(t)u^{[1]}(t) \in L^1_{loc}(\mathbf{R}_+) \ . \tag{11}$$

In fact this v coincides with Hu in Definition 1.

The equation $Hu = \lambda u$ for $\lambda \in \mathbf{C}$ is interpreted as the system of integral equations

$$\begin{aligned} u(t) &= u(0) + \int_0^t u'(s)ds \ ; \\ u'(t) &= u'(0) + u(0)(Q(t) + P(t) - \lambda t) \\ &\quad + \int_0^t \{Q(t) - Q(y) - \lambda(t-y) + P(t) - P(y)\}u'(y)dy \ , \end{aligned} \tag{12}$$

where $P(t) := \int_0^t p(s)ds$, or of differential equations

$$\frac{d}{dt}\begin{bmatrix} u(t) \\ u^{[1]}(t) \end{bmatrix} = \begin{bmatrix} Q(t) & 1 \\ \lambda - Q^2(t) - p(t) & -Q(t) \end{bmatrix}\begin{bmatrix} u(t) \\ u^{[1]}(t) \end{bmatrix} \ , \tag{13}$$

with a given initial condition, both (12) and (13) being uniquely solvable. Moreover, the Green's formula is valid in the sense that for any $u_1, u_2 \in \mathcal{C}$ and $0 \le a < b < \infty$, one has

$$\int_a^b \{(Hu_1)(t)u_2(t) - u_1(t)(Hu_2)(t)\}dt = [u_1,u_2](t)|_{t=a}^{t=b} \ , \tag{14}$$

where

$$[u_1,u_2](t) := u_1(t)u_2'(t) - u_1'(t)u_2(t) = u_1(t)u_2^{[1]}(t) - u^{[1]}(t)u_2(t) \ . \tag{15}$$

With these interpretations, it can be verified that H can be treated quite similarly as the classical Sturm-Liouville operator. For example, if we define two spaces

$$\mathcal{D}_0 := \{u \in \mathcal{C} \cap L^2(\mathbf{R}_+);\ u(0) = 0\ , \quad Hu \in L^2(\mathbf{R}_+)\} \tag{16}$$

and

$$\mathcal{D}_0^S := \{u \in \mathcal{D}_0;\ \lim_{t\to\infty}[u,v](t) = 0 \text{ for any } v \in \mathcal{D}_0\}\ , \tag{17}$$

and if we let

$$H_0 = H|_{\mathcal{D}_0}\ ; \quad H_0^S = H|_{\mathcal{D}_0^S}\ , \tag{18}$$

then we can prove that $H_0^* = H_0^S$ and $(H_0^S)^* = H_0$ hold in exactly the same way as the proof of Theorem 10.11 of [15]. (From that argument, it follows in particular that $\mathcal{D}_0^S$ is dense in $L^2(\mathbf{R}_+)$.) Thus H_0^S is a closed symmetric operator. Moreover, Weyl-Titchmarsh theory is valid also for H, and it holds that H_0^S is self-adjoint if and only if H is of limit point type at $+\infty$, which is true if and only if for some $\lambda \in \mathbf{C}$, the equation $Hu = \lambda u$ has a solution which is *not* square integrable near $+\infty$.

Taking $Q(t) = cX_\omega(t)$ in the definition of H_0^S above, we can realize the stochastic Airy operator as a closed symmetric operator in $L^2(\mathbf{R}_+)$.

We shall denote by $H_0(\omega)$ and $H_0^S(\omega)$ the operators defined by (18) with this choice of $Q(t)$. Also we denote by $H(\omega)$ the expression (9) itself with the same choice of $Q(t)$.

Remark 1 *As is pointed out by Edelman and Sutton [4] and Bloemendal [1], we can transform H in (9) to a classical Sturm-Liouville operator*

$$Lf = -\frac{d^2}{dt^2}f + pf - 2Q\frac{d}{dt}f - Q^2 f$$

by letting $u = f\phi$, with $\phi(t) = \exp\Big(\int_0^t Q(x)dx\Big)$. This makes it much clearer that Weyl-Titchmarsh theory is still valid for our generalized Sturm-Liouville operator H.

For a recent systematic treatment of generalized Sturm-Liouville operator including those described in §2, see for example [3].

3 Almost sure self-adjointness of the stochastic Airy operator.

In this section, we construct a self-adjoint operator A_ω which is associated with the quadratic form $\mathcal{E}_\omega$ defined in (6), and show that $A_\omega = H_0^S(\omega)$. All necessary assertions concerning $\mathcal{E}_\omega$ are deduced from the lemma below.

Lemma 1 *If $\{X_\omega(t)\}$ is the fractional Brownian motion with Hurst parameter $h \in (0,1)$, then with probability one, $a_\omega'(t) = \mathcal{O}(\sqrt{\log t})$ and $X_\omega(t) - a_\omega(t) = \mathcal{O}(\sqrt{\log t})$ as $t \to \infty$, where $a_\omega(t)$ is defined by (8).*

Proof. Let $n \le t \le n-1$. We then have

$$|a_\omega'(t)| = |X_\omega(t+1) - X_\omega(t)| \le |X_\omega(t+1) - X_\omega(n+1)| + |X_\omega(n+1) - X_\omega(n)| + |X_\omega(t) - X_\omega(n)|\ ,$$

and hence

$$\sup_{n\le t\le n+1} |a_\omega'(t)| \le 2 \sup_{0\le s\le 1} |X_\omega(n+s) - X_\omega(n)| + \sup_{0\le s\le 1} |X_\omega(n+1+s) - X_\omega(n+1)|\ .$$

Since the process $\{X_\omega(t+T)-X_\omega(T)\}_{t\ge 0}$ has the same law as $\{X_\omega(t)\}_{t\ge 0}$ for each $T\ge 0$, we see that for $b>0$ and $n\ge 1$,

$$\begin{aligned}
& \mathbf{P}\Big(\sup_{n\le t\le n+1}|a'_\omega(t)|\ge b\sqrt{\log n}) \\
\le\ & \mathbf{P}\Big(\sup_{0\le s\le 1}|X_\omega(n+s)-X_\omega(n)|\ge \frac{b}{3}\sqrt{\log n}\Big)+\mathbf{P}\Big(\sup_{0\le s\le 1}|X_\omega(n+1+s)-X_\omega(n+1)|\ge \frac{b}{3}\sqrt{\log n}\Big) \\
=\ & 2\mathbf{P}\Big(\sup_{0\le s\le 1}|X_\omega(s)|\ge \frac{b}{3}\sqrt{\log n}\Big)\ .
\end{aligned}$$

Now it is known (see Theorem 4.1 of [10]) that for the fractional Brownian motion with Hurst parameter $h\in(0,1)$ the estimate

$$\lim_{x\to\infty}x^{-2}\log\mathbf{P}\Big(\sup_{0\le s\le 1}X_\omega(s)\ge x\Big)=-\frac{1}{2}$$

holds. Hence for any $\varepsilon\in(0,1)$, we can choose a $K_\varepsilon>0$ such that

$$\mathbf{P}\Big(\sup_{0\le s\le 1}|X_\omega(t)|\ge x\Big)\le K_\varepsilon e^{-(1-\varepsilon)x^2/2}\ .$$

Hence for b greater thatn $\sqrt{2/(1-\varepsilon)}$, we have

$$\begin{aligned}
\sum_{n\ge 1}\mathbf{P}\Big(\sup_{n\le t\le n+1}|a'_\omega(t)|\ge b\sqrt{\log n}\Big) &\le \sum_{n\ge 1}2K_\varepsilon\exp\Big[-\frac{1-\varepsilon}{2}b^2\log n\Big] \\
&= 2K_\varepsilon\sum_{n\ge 1}n^{-(1-\varepsilon)b^2/2}<\infty\ ,
\end{aligned}$$

and hence by Borel-Cantelli lemma, for $\mathbf{P}$-almost all $\omega\in\Omega$, we can choose a constant C_ω so that

$$\sup_{n\le t\le n+1}|a'_\omega(t)|\le C_\omega\sqrt{\log n}\quad (n\ge 1)\ .$$

As to the assertion for $X_\omega(t)-a_\omega(t)$, we first note

$$\sup_{n\le t\le n+1}|X_\omega(t)-a_\omega(t)|=\sup_{n\le t\le n+1}\left|\int_0^1(X_\omega(t+s)-X_\omega(t))ds\right|\le\sup_{n\le t\le n+1}\sup_{0\le s\le 1}|X_\omega(t+s)-X_\omega(t)|$$

and

$$\begin{aligned}
& |X_\omega(t+s)-X_\omega(t)| \\
\le\ & \begin{cases} |X_\omega(t)-X_\omega(n)|+|X_\omega(t+s)-X_\omega(n)| & (0\le s\le n+1-t) \\ |X_\omega(t+s)-X_\omega(n+1)|+|X_\omega(n+1)-X_\omega(n)|+|X_\omega(t)-X_\omega(n)| & (n+1-t\le s\le 1)\ . \end{cases}
\end{aligned}$$

Then as in the case of $a'_\omega(t)$, we again have

$$\sup_{n\le t\le n+1}|X_\omega(t)-a_\omega(t)|\le 2\sup_{0\le s\le 1}|X_\omega(n+s)-X_\omega(n)|+\sup_{0\le s\le 1}|X_\omega(n+1+s)-X_\omega(n+1)|\ ,$$

and the rest of the proof is the same as the preceding argument.

The following proposition summarizes all the properties of $\mathcal{E}_\omega$ which we need.

Proposition 1 *Fix an ω for which the conclusion of Lemma 1 holds. Then under the assumption of Theorem 2, we have that*

(i) *$\mathcal{E}_\omega(u,v)$ is well defined for all $u,v \in L$;*

(ii) *$\mathcal{E}_\omega$ is lower semi-bounded;*

(iii) *$\mathcal{E}_\omega$ is closed;*

(iv) *$\mathcal{E}_\omega$ is completely continuous in the sense that any L^2-bounded sequence $\{u_n\}$ in L such that $\sup_n \mathcal{E}_\omega(u_n,u_n) < \infty$ contains an L^2-convergent subsequence.*

Proof. By Lemma 1, we have $p(t)+ca'_\omega(t) = \mathcal{O}(1+p(t))$ and $X_\omega(t)-a_\omega(t) = o(\sqrt{1+p(t)})$ as $t \to \infty$. In particular for any $\varepsilon \in (0,1/c)$, one can choose a constant $C_1 = C_1(\omega,\varepsilon)$ so that $|X_\omega(t)-a_\omega(t)| \le \varepsilon\sqrt{C_1+p(t)}$ holds for all $t \ge 0$. This together with Schwarz inequality shows that the integrals in (6) are absolutely convergent for all $u,v \in L$, and hence $\mathcal{E}_\omega$ is well defined on L. To verify assertion (ii), we write

$$\begin{aligned}\mathcal{E}_\omega(u,u) &= \int_0^\infty |u'(t)|^2 dt + \int_0^\infty (p(t)+ca'_\omega(t))|u(t)|^2 dt \\ &\quad -c\int_0^\infty (X_\omega(t)-a_\omega(t))\Big(u(t)\overline{u'(t)} + u'(t)\overline{u(t)}\Big)dt \end{aligned} \tag{19}$$

for $u \in L$. Noting $2|xy| \le |x|^2+|y|^2$, we see that the third integral on the right hand side of (19) is bounded in absolute value by

$$2\varepsilon c\int_0^\infty \sqrt{C_1+p(t)}|u(t)||u'(t)|dt \le \varepsilon c\Big\{\int_0^\infty (C_1+p(t))|u(t)|^2 dt + \int_0^\infty |u'(t)|^2 dt\Big\}\ ,$$

which is finite. Hence we have

$$\begin{aligned}\mathcal{E}_\omega(u,u) &\ge (1-\varepsilon c)\int_0^\infty |u'(t)|^2 dt - \varepsilon c C_1\int_0^\infty |u'(t)|^2 dt \\ &\quad + \int_0^\infty \{(1-\varepsilon c)p(t) + ca'_\omega(t)\}|u(t)|^2 dt \end{aligned} \tag{20}$$

If we further take a $\delta \in (0, 1-\varepsilon c)$, then there is a constant $C_2 = C_2(\omega, p(\cdot), c, \varepsilon, \delta)$ such that

$$\inf_{t\ge 0}\{(1-\varepsilon c-\delta)p(t) + ca'_\omega(t)\} \ge -C_2\ . \tag{21}$$

Hence letting $C = \varepsilon c C_1 + C_2$, we obtain the lower bound

$$\begin{aligned}\mathcal{E}_\omega(u,u) &\ge (1-\varepsilon c)\int_0^\infty |u'(t)|^2 dt + \delta\int_0^\infty p(t)|u(t)|^2 dt - C\int_0^\infty |u(t)|^2 dt \\ &\ge -C\int_0^\infty |u(t)|^2 dt = -C(u,u)\ , \end{aligned} \tag{22}$$

which proves the assertion (ii).

Now fix a constant $\gamma > C$ and define the norm $\|\cdot\|_\gamma$ on L by $\|u\|_\gamma^2 = \mathcal{E}_\omega(u,u) + \gamma(u,u)$. Then by (22),

$$\|u\|_\gamma^2 \ge (1-\varepsilon c)\int_0^\infty |u'(t)|^2 dt + \delta\int_0^\infty p(t)|u(t)|^2 dt + (\gamma - C)\int_0^\infty |u(t)|^2 dt \tag{23}$$

holds for $u \in L$. To show that L is complete under the norm $\|\cdot\|_\gamma$, let $\{u_n\} \subset L$ be a Cauchy sequence with respect to this norm. For this sequence, we have from (23)

$$\lim_{n,m\to\infty} \|u_n - u_m\|_2 = 0\ , \tag{24}$$

$$\lim_{n,m\to\infty} \|u'_n - u'_m\|_2 = 0, \tag{25}$$

and

$$\lim_{n,m\to\infty}\int_0^\infty p(t)|u_n(t)-u_m(t)|^2dt = 0\ . \tag{26}$$

By (24) and (25), there are $u, v \in L^2(\mathbf{R}_+)$ such that $\|u_n - u\|_2 \to 0$ and $\|u'_n - v\|_2 \to 0$ as $n \to \infty$. In particular we have $\int_0^x |u'_n(t) - v(t)|dt \to 0$ for any $x > 0$. Hence we can let $n \to \infty$ in the equality $u_n(x) = \int_0^x u'_n(t)dt$ and we see that $\{u_n(x)\}_n$ converges pointwise to a continuous function which must coincide almost everywhere with $u(t)$, namely $u(x) = \int_0^x v(t)dt$ for almost every x. Thus $u(t)$ is absolutely continuous, $u(0) = 0$, and $u'(t) = v(t) \in L^2(\mathbf{R}_+)$. By (26) and Fatou's lemma we get

$$\int_0^\infty p(t)|u(t)|^2dt \le \liminf_{n\to\infty}\int_0^\infty p(t)|u_n(t)|^2dt < \infty\ ,$$

so that $u \in L$, and

$$\lim_{n\to\infty}\int_0^\infty p(t)|u_n(t)-u(t)|^2dt = 0\ .$$

Since we can choose a constant K_ω such that

$$\mathcal{E}_\omega(u,u) \le K_\omega\left\{\int_0^\infty |u(t)|^2dt + \int_0^\infty |u'(t)|^2dt + \int_0^\infty p(t)|u(t)|^2dt\right\}$$

in the same way as we proved (23), we finally obtain $\|u_n - u\|_\gamma \to 0$, and hence L is complete under $\|\cdot\|_\gamma$.

If $\{u_n\} \subset L$ is a $\|\cdot\|_\gamma$-bounded sequence, then there are constants M_1, M_2, M_3 such that

$$\int_0^\infty |u_n(t)|^2dt \le M_1,\ \int_0^\infty |u'_n(t)|^2dt \le M_2,\ \int_0^\infty p(t)|u_n(t)|^2dt \le M_3 \tag{27}$$

hold for all $n \ge 1$. By the relation $u_n(x)^2 = 2\int_0^x u_n(t)u'_n(t)dt$, we have

$$|u_n(x)|^2 \le 2\int_0^\infty |u_n(t)||u'_n(t)|dt \le 2\sqrt{\int_0^\infty |u_n(t)|^2dt}\sqrt{\int_0^\infty |u'_n(t)|^2dt} \le \sqrt{M_1M_2}\ ,$$

so that $\{u_n(x)\}$ is uniformly bounded. Since for any $0 \le x < y$,

$$|u_n(x) - u_n(y)| \le \int_x^y |u'_n(t)| \le \sqrt{\int_x^y 1dt}\sqrt{\int_0^\infty |u'_n(t)|^2dt} \le \sqrt{M_2}|y-x|$$

holds for all $n \ge 1$, the sequence $\{u_n\}$ is uniformly equi- continuous. Hence it contains a subsequence $\{u_{n'}(x)\}$ which converges uniformly on every compact subset of $[0,\infty)$ to a continuous function $u(t)$. Now for sufficiently large $X > 0$ it holds that

$$\int_X^\infty |u_n(t)|^2dt \le \frac{1}{\inf_{s\ge X} p(s)}\int_X^\infty p(t)|u_n(t)|^2dt \le \frac{M_3}{\inf_{s\ge X} p(s)}\ ,$$

and so

$$\begin{aligned}
& \int_0^\infty |u_{n'}(t) - u(t)|^2dt \\
\le\ & \int_0^X |u_{n'}(t) - u(t)|^2dt + \int_X^\infty |u(t)|^2dt + \int_X^\infty |u_{n'}(t)|^2dt \\
\le\ & \int_0^X |u_{n'}(t) - u(t)|^2dt + \liminf_{m'\to\infty}\int_X^\infty |u_{m'}(t)|^2dt + \int_X^\infty |u_{n'}(t)|^2dt \\
\le\ & \int_0^X |u_{n'}(t) - u(t)|^2dt + \frac{2M_3}{\inf_{s\ge X} p(s)}\ .
\end{aligned}$$

By letting $n' \to \infty$ first, and then $X \to \infty$, we obtain $\lim_{n'\to\infty} \|u_{n'} - u\|_2 = 0$.

We can now apply general theory, to obtain a self-adjoint operator A_ω with domain $D(A_\omega) \subset L$ such that $(A_\omega u, v) = \mathcal{E}_\omega(u, v)$ for all $u, v \in D(A_\omega)$. In particular, $\psi \in D(A_\omega)$ if and only if there is a $\varphi \in L^2(\mathbf{R}_+)$ such that $\mathcal{E}_\omega(u, \psi) = (u, \varphi)$ for all $u \in L$ (see [12], Theorem VIII.15). Moreover, the assertion (iv) above implies that A_ω has purely discrete spectrum (see [13], Theorem XIII.64). We shall show that $A_\omega = H_0^S(\omega)$ holds for every fixed ω for which the conclusion of Lemma 1 is valid. Let us begin by proving the following assertion.

Proposition 2 *The relation $H_0^S(\omega) \subset A_\omega \subset H_0(\omega)$ holds.*

Proof. We prove $A_\omega \subset H_0(\omega)$ only. The other inclusion relation follows by taking the adjoint.
From the definition of the quadratic form $\mathcal{E}_\omega$, it is easy to see that $\psi \in D(A_\omega)$ if and only if there is a $\varphi \in L^2(\mathbf{R}_+)$ such that

$$\begin{aligned} & \int_0^\infty u'(t)\{\overline{\psi'(t)} - c(X_\omega(t) - a_\omega(t))\overline{\psi(t)}\}dt \\ = & \int_0^\infty u(t)\{\overline{\varphi(t)} - (p(t) + ca'_\omega(t))\overline{\psi(t)} + c(X_\omega(t) - a_\omega(t))\overline{\psi'(t)}\}dt \end{aligned} \tag{28}$$

holds for all $u \in L$. In particular, if $\psi \in D(A_\omega)$, then the relation (28) holds for all $u \in L$ with compact support. Then by Lemma 2 below, we can conclude that

$$\psi'(t) - cX_\omega(t) + ca_\omega(t)\psi(t) = \psi^{[1]}(t) + ca_\omega(t)\psi(t)$$

is absolutely continuous and

$$-(\psi^{[1]}(t) + ca_\omega(t)\psi(t))' = \varphi(t) - (p(t) + ca'_\omega(t))\psi(t) + c(X_\omega(t) - a_\omega(t))\psi'(t) \ ,$$

namely

$$\varphi(t) = -(\psi^{[1]}(t))' - cX_\omega(t)\psi^{[1]}(t) + (p(t) - c^2X_\omega(t)^2)\psi(t) \ . \tag{29}$$

This shows that $\psi \in \mathcal{D}_0(\omega)$ and $H_0(\omega)\psi = \varphi$.

Lemma 2 *Let $\alpha(t)$ be locally bounded and measurable, $\gamma(t)$ be locally integrable on $[0, \infty)$. If*

$$\int_0^\infty u'(t)\alpha(t)dt = \int_0^\infty u(t)\gamma(t)dt \tag{30}$$

holds for all $u \in L$ with compact support, then $\alpha(t)$ is absolutely continuous and $-\alpha'(t) = \gamma(t)$ holds almost everywhere on $[0, \infty)$.

Proof. For a given $T > 0$, it is easy to see that $u \in L$ has its support in $[0, T]$ if and only if there is a $v \in L^2(\mathbf{R}_+)$ such that

$$u(t) = \int_0^{t\wedge T} (v(s) - \langle v\rangle_T)ds \tag{31}$$

holds, where we have set $\langle v\rangle_T = \frac{1}{T}\int_0^T v(s)ds$. Hence (30) becomes

$$\int_0^T (v(t) - \langle v\rangle_T)\alpha(t)dt = \int_0^T \left\{\int_0^t (v(s) - \langle v\rangle_T)ds\right\}\gamma(t)dt \ ,$$

which can be rewritten in the following form

$$\int_0^T v(t)(\alpha(t) - \langle\alpha\rangle_T)dt = \int_0^T v(t)(C(t) - \langle C\rangle_T)dt \ , \tag{32}$$

where $C(t) := \int_t^T \gamma(s)ds$. Since v runs through all of $L^2([0,T])$, we have $\alpha(t) - \langle\alpha\rangle_T = C(t) - \langle C\rangle_T$ almost everywhere on $[0,T]$, so that $\alpha'(t) = -\gamma(t)$. Since $T > 0$ is arbitrary, we arrive at the desired conclusion.

To the best of the author's knowledge, it was Fukushima and Nakao [5] who first treated one-dimensional Schrödinger operator with singular potential including Gaussian white noise. In fact, they considered the operator $H_h := -d^2/dt^2 + h'(t)$ on a finite interval $[a,b]$ under the Dirichlet boundary condition $u(a) = u(b) = 0$, where $h(t)$ is a bounded Borel function on $[a,b]$. They defined it as the self-adjoint operator A_h associated to the quadratic form

$$\mathcal{E}_h(u,v) = \int_a^b u'(t)\overline{v(t)}dt - \int_a^b h(t)\{u(t)\overline{v(t)}\}'dt \ , \tag{33}$$

which is defined on

$$H_0^1(a,b) = \{u;\ u(\cdot) \text{ is absolutely continuous on } [a,b],\ u'(\cdot) \in L^2([a,b]) \ , \text{ and } u(a) = u(b) = 0\} \tag{34}$$

and is closed, lower semi-bounded and completely continuous. In a similar manner as Lemma 2 above, it can be verified that a function $\psi \in L^2([a,b])$ belongs to the domain of A_h if and only if ψ and $\psi^{[1]} := \psi' - h\psi$ are absolutely continuous on $[a,b]$, $\psi(a) = \psi(b) = 0$ and $\varphi := -(\psi^{[1]})' - h\psi^{[1]} - h^2\psi \in L^2([a,b])$. And in this case, we have $\varphi = A_h\psi$. Thus the operator A_h of Fukushima and Nakao coincides with the operator described in §2.

If we introduce Prüfer variables $r_\lambda(t)$ and $\theta_\lambda(t)$ by

$$\psi(t) = r_\lambda(t)\sin\theta_\lambda(t) \ , \quad \psi^{[1]}(t) = r_\lambda(t)\cos\theta_\lambda(t) \ , \tag{35}$$

where ψ is a solution to $H_h\psi = \lambda\psi$, then in exactly the same way as the classical Sturm-Liouville theory, one can prove the following oscillation theorem.

Proposition 3 *$\theta_\lambda(t)$ can be defined as a jointly continuous function of (t,λ), which is strictly increasing in λ for each fixed t, and which satisfies $d\theta_\lambda(t)/dt = 1$ whenever $\theta_\lambda(t) \equiv 0$ (mod π). If $\lambda_1 < \lambda_2 < \cdots$ are the eigenvalues of A_h, then the eigenfunction $\psi_h(t)$ belonging to the k-th eigenvalue λ_k has exactly $k-1$ zeros in (a,b). In other words, there exist $a < t_1 < \cdots < t_{k-1} < b$ such that $\theta_{\lambda_k}(t_j) = j\pi$ ($j = 1,\ldots,k-1$).*

Let us return to the proof of Theorem 2. Fix an ω for which the conclusion of Lemma 1 is true, and let $\lambda_1(\omega)$ and $\psi_{1,\omega}(t)$ be the lowest eigenvalue of A_ω and its eigenfunction. We shall show that $\psi_{1,\omega}(t)$ has no zeros on $(0,\infty)$, and apply the following theorem of Hartman [6], to conclude that $H(\omega)$ is of limit-point type at $+\infty$ and that consequently $H_0^S(\omega)$ is self-adjoint. We note that Hartman's theorem is valid also for our generalized Sturm-Liouville operator.

Proposition 4 (Part (i) of Theorem in [6]) *Let $H = H(p,Q)$ be a generalized Sturm-Liouville operator as in (9). If for some real number μ, the equation $H\varphi = \mu\varphi$ has a solution ψ with only finitely many zeros on $[0,\infty)$, then H is of limit-point type at $+\infty$.*

Let L_0 be the subspace of L consisting of all compactly supported functions belonging to L. Then we have

$$\begin{aligned} \lambda_1(\omega) &= \inf\{\mathcal{E}_\omega(u,u);\ u \in L \ , \quad \|u\|_2 = 1\} \\ &\le \inf\{\mathcal{E}_\omega;\ u \in L_0 \ , \quad \|u\|_2 = 1\} \\ &= \lim_{l\to\infty} \inf\{\mathcal{E}_\omega;\ u \in L_0 \ , \quad \|u\|_2 = 1 \ , \quad \text{supp } u \subset [0,l]\} \ . \end{aligned} \tag{36}$$

If we denote by $\lambda_1^l(\omega)$ the infimum appearing on the right hand side of (36), then $\lambda_1^l(\omega)$ is nothing but the lowest eigenvalue of $H(\omega)$ on $[0,l]$ considered under the Dirichlet boundary condition $u(0) = u(l) = 0$. Hence its eigenfunction $\psi_{1,\omega}^l(t)$ has no zeros on $(0,l)$.

Now suppose $\psi_{1,\omega}(t)$ has zeros on $(0,\infty)$ and let l be the smallest among them, so that $\psi_{1,\omega}(t) \neq 0$ for $0 < t < l$ and $\psi_{1,\omega}(l) = 0$. This shows that the function $\psi_{1,\omega}(t)$, considered on $0 \leq t \leq l$, is the first eigenfunction of $H(\omega)$ on $[0,l]$ with the Dirichlet boundary condition at both endpoints, namely $\lambda_1(\omega) = \lambda_1^l(\omega)$. If $\theta_\lambda(t)$, $\lambda = \lambda_1(\omega) = \lambda_1^l(\omega)$, is the Prüfer variable for $\psi_{1,\omega}(t)$, then we have $\theta_\lambda(0) = 0$, $0 < \theta_\lambda(t) < \pi$ for $0 < t < l$, and $\theta_\lambda(l) = \pi$. On the other hand, if we take $l' > l$ and if we let $\theta_{\lambda'}(t)$, $\lambda' := \lambda_1^{l'}(\omega)$, be the Prüfer variable for the first eigenfunction $\psi_{1,\omega}^{l'}(t)$ of $H(\omega)$ on $[0,l']$ with the Dirichlet boundary condition at both endpoints, then we have $\theta_{\lambda'}(0) = 0$, $0 < \theta_{\lambda'}(t) < \pi$ for $0 < t < l'$, and $\theta_{\lambda'}(l') = \pi$. Comparing this behavior of $\theta_{\lambda'}(t)$ to that of $\theta_\lambda(t)$ above, we must conclude $\lambda' < \lambda$ from the strict monotonicity of $\theta_\lambda(t)$ with respect to λ, namely $\lambda_1^{l'}(\omega) < \lambda_1^l(\omega) = \lambda_1(\omega)$ for any $l' > l$, contradicting (36). The proof of Theorem 2 is now complete.

Remark 2 *(i) For $k \geq 2$, let $\lambda_k(\omega)$ and $\psi_{k,\omega}(t)$ be the k-th eigenvalue and eigenfunction of $A_\omega = H_0^S(\omega)$, where the conclusion of Lemma 1 is valid for ω. Then $\psi_{k,\omega}(t)$ has exactly $k-1$ zeros on $(0,\infty)$. This can be proved in the same way as above if we note the following min-max characterization of $\lambda_k(\omega)$:*

$$\lambda_k(\omega) = \inf_M \sup\{\mathcal{E}_\omega(u,u);\ u \in M,\ \|u\|_2 = 1\}\ , \tag{37}$$

where M runs through all k-dimensional linear subspaces of L.
(ii) Once we have verified that $\psi_{k,\omega}(t)$ has only finitely many zeros for all k, it is now obvious that for any real λ, all non-trivial solution of $H(\omega)u = \lambda u$ has only finitely many zeros on $(0,\infty)$.

Thus from Corollary in [6], we see that $H(\omega)$, considered under the boundary condition $u(0)\cos\theta + u'(0)\sin\theta = 0$, also defines a semi-bounded self-adjoint operator with purely discrete spectrum. The k-th eigenfunction of this operator has exactly $k-1$ zeros on $(0,\infty)$.

4 Remarks and applications.

4.1 Explosion of the solution of a stochastic Riccati equation.

Let H be a generalized Sturm-Liouville operator described in §2, and let $u(t)$ be a non-trivial solution of the equation $Hu = \lambda u$. If we let $z_\lambda(t) = u'(t)/u(t)$, then it is easily seen that it satisfies

$$z_\lambda(t) - z_\lambda(s) = Q(t) - Q(s) + \int_s^t \{p(y) - \lambda - z_\lambda(y)^2\}dy \tag{38}$$

on every interval on which $u(t)$ never vanishes, and that $z_\lambda(\tau \pm 0) = \pm\infty$ whenever $u(\tau) = 0$.

If $Q(t) = cX_\omega(t)$, $X_\omega(t)$ being a sample function of the fractional Brownian motion of Hurst parameter $h \in (0,1)$, then (38) defines a stochastic process $\{z_{\lambda,\omega}(t)\}$ with state space $[-\infty,+\infty]$. If, in addition, $p(t)$ satisfies the condition of Theorem 2, then from Remark 2, we obtain the following result.

Proposition 5 *For $\mathbf{P}$-almost all $\omega \in \Omega$, it holds that $\{z_{\lambda,\omega}(t)\}$ explodes to $-\infty$ finitely often for any real λ and for any initial value $z(0)$ including $+\infty$.*

4.2 Schrödinger operator with Gaussian white noise potential under a uniform electric field.

In [9], the present author considered a random Schrödinger operator on $[0,\infty)$ of the form

$$H^\theta_{\kappa,F}(\omega) = -\frac{d^2}{dt^2} + \kappa B'_\omega(t) - Ft\ , \tag{39}$$

under the boundary condition $u(0)\cos\theta + u'(0)\sin\theta = 0$ at the origin. Here $\{B_\omega(t)\}$ is the standard Brownian motion as before and the constants κ and F are strictly positive. The following result was obtained.

Theorem 3 ([9]) *For any fixed value of $F > 0$, $\kappa > 0$ and $\theta \in [0, \pi)$, the operator $H^{\theta}_{\kappa,F}(\omega)$ is, with probability one, self-adjoint, its spectrum being $(-\infty, \infty)$. When $0 < F < \kappa^2/2$, the spectrum of $H^{\theta}_{\kappa,F}(\omega)$ is almost surely pure point, while when $F \geq \kappa^2/2$, it is purely singular continuous.*

Combining Theorem 3, together with its proof in [9], and Theorem 2 of the present work, we obtain the following result.

Theorem 4 *The random Schrödinger operator*

$$H_{\kappa,F}(\omega) = -\frac{d^2}{dt^2} + \kappa B'_{\omega}(t) - Ft$$

on the whole space $\mathbf{R}$ is almost surely a self-adjoint operator in the Hilbert space $L^2(\mathbf{R})$, and its spectrum as a set is $(-\infty, \infty)$. $H_{\kappa,F}(\omega)$ has pure point spectrum with probability one when $0 < F < \kappa^2/2$, while it has purely singular continuous spectrum if $F \geq \kappa^2/2$.

Proof. Theorems 2 and 3 tells us that $H_{\kappa,F}(\omega)$ is almost surely of limit point type both at $+\infty$ and $-\infty$, so that it defines a unique self-adjoint operator almost surely. We know that Condition K in [9] holds for $H_{\kappa,F}(\omega)$. Moreover it was proved in [9] that when $0 < F < \kappa^2/2$, the condition of Theorem 4 (ii) there is satisfied, while when $F \geq \kappa^2/2$, that of Theorem 4 (vi) holds. On the other hand, by Theorem 2, Remark 2 and Lemma in [6], we see that with probability one, the equation $H_{\kappa,F}(\omega)u = \lambda u$ has a non-trivial solution which is square integrable near $-\infty$ for every real λ. Hence accordingly as $0 < F < \kappa^2/2$ or $F \geq \kappa^2/2$, $H_{\kappa,F}(\omega)$ satisfies the condition (i) or (v) of Theorem 4 of [9], and the desired conclusion follows.

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