New Proofs of Some Basic Theorems on Stationary Point Processes

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Summary– We give new proofs of three basic theorems on stationary point processes on the real line – theorems of Khintchine, Korolyuk, and Dobrushin. Moreover we give a direct construction of the Palm measure for a class of point processes which includes stationary ones as special cases.

1 Introduction.

The purpose of this note is to give new proofs, based on a same simple idea, to some basic theorems on stationary point processes on the real line $\mathbf{R}$, as stated in standard treatises on point processes such as Daley and Vere-Jones (see §3.3 of [3]).

To begin with, let us introduce necessary definitions and notation. By M_p, we denote the set of all integer-valued Radon measures on $\mathbf{R}$. Namely M_p is the totality of all measures $N(dx)$ on $\mathbf{R}$ such that for any bounded Borel set B, $N(B)$ is a non-negative integer. Let us call any such measure a *counting measure*. For a counting measure $N \in M_p$, let us define

$$X(t) := N((0,t]) \quad (t \geq 0), \qquad := -N((t,0)) \quad (t < 0). \tag{1}$$

Then the function $X(t)$ is right-continuous, integer-valued, locally bounded and non-decreasing. Hence $X(t)$ is piecewise constant on $\mathbf{R}$ and the set Δ, finite or countably infinite, of its points of discontinuity has no accumulation points other than $\pm\infty$. Thus the points in Δ can be ordered as

$$\cdots < x_{-1} < x_0 \leq 0 < x_1 < x_2 < \cdots,$$

so that if we let $m_n := X(x_n) - X(x_n - 0)$, then $N(dx)$ can be represented as

$$N(dx) = \sum_n m_n \delta_{x_n}(dx), \tag{2}$$

where δ_a denotes the unit mass placed at a. Each m_n is a positive integer and is called the *multiplicity* of the point x_n. In general, either $N([0,\infty))$ or $N((-\infty,0))$ can be finite, in

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which case either $\{x_n\}_{n>0}$ or $\{x_n\}_{n\le 0}$ is a finite sequence. If in the former [resp. latter] case $\{x_n\}_{n>0}$ [resp. $\{x_n\}_{n\le 0}$] terminates with x_ν, then we will set $x_n = \infty$ [resp. $x_n = -\infty$] for $n > \nu$ [resp. $n < \nu$]. When $m_n = 1$ for all n such that $x_n \ne \pm\infty$, the counting measure N is said to be *simple*. For each $N \in M_p$ with representation (2), let us associate a simple counting measure N^* defined by

$$N^*(dx) = \sum_n \delta_{x_n}(dx) \ . \tag{3}$$

In order to make M_p a measurable space, we define $\mathcal{M}_p$ to be the σ-algebra of subsets of M_p generated by all mappings of the form

$$M_p \ni N \mapsto N(B) \in [0, \infty] \tag{4}$$

for all Borel sets $B \subset \mathbf{R}$. Then we see that x_n, m_n and N^* are all measurable functions of N, as the following lemma shows.

Lemma 1 *(i) The set*

$$C := \{N \in M_p : \ N((-\infty, 0]) = N((0, \infty)) = \infty\} = \{N \in M_p : \ x_n \textit{is finite for all } n\}$$

belongs to $\mathcal{M}_p$.

(ii) For each integer n, x_n and m_n are $\mathcal{M}_p$-measurable functions of N.

(iii) The mapping $M_p \ni N \mapsto N^* \in M_p$ is $\mathcal{M}_p/\mathcal{M}_p$-measurable.

Proof. (i) The assertion is obvious from the definition of $\mathcal{M}_p$, since we can write

$$C = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{N \in M_p : \ N((-n, 0]) > k, \ N((0, n]) > k\} \ .$$

(ii) The measurability of x_1 follows from the relation

$$\{N \in M_p : \ x_1 > t\} = \{N \in M_p : \ N((0, t]) = 0\} \ ,$$

which holds for all $t \ge 0$. Now for each $k \ge 1$, define

$$x_1^{(k)} := \sum_{j=1}^{\infty} \frac{j}{2^n} \mathbf{1}_{((j-1)/2^n, j/2^n]}(x_1) + \infty \cdot \mathbf{1}_{\{x_1 = \infty\}} \ .$$

Then we see that $x_1^{(k)}$ is measurable in N and that $x_1^{(k)} \searrow x_1$ as $k \to \infty$. By the right-continuity of $X(t) = N((0, t])$ at $t > 0$, we have, as $k \to \infty$,

$$\mathbf{1}_{\{x_1 < \infty\}} \cdot X(x_1^{(k)}) = \sum_{j=1}^{\infty} \mathbf{1}_{((j-1)/2^n, j/2^n]}(x_1) X(\frac{j}{2^k}) \longrightarrow X(x_1) = m_1 \ ,$$

which shows the measurability of m_1 in N.

Next let $\tilde{X}(t) := X(t) - X(t \wedge x_1)$. This is measurable in N for all $t \geq 0$, since

$$X(t \wedge x_1) = X(t)\mathbf{1}_{\{x_1 \geq t\}} + X(x_1)\mathbf{1}_{\{x_1 < t\}} \ .$$

If we apply the above argument to $\tilde{X}(t)$ instead of $X(t)$, we can verify the measurability of x_2 and m_2 in N, and the argument can be iterated to give the measurability of all x_n and m_n.

(iii) For each $j = 0, 1, 2, \ldots$ and $t > 0$, the sets

$$\{N \in M_p : \ N^*((0,t]) = j\} = \{N \in M_p : \ x_j \leq t < x_{j+1}\}$$

and

$$\{N \in M_p : \ N^*((-t,0]) = j\} = \{N \in M_p : \ x_{-j} \leq t < x_{-j+1}\}$$

belong to $\mathcal{M}_p$. Now for each $n \geq 1$, let $\mathcal{G}_n$ be the class of all Borel subsets B of $[-n, n]$ such that the mapping

$$M_p \ni N \mapsto N^*(B) \in [0, \infty) \tag{5}$$

is measurable. Then $\mathcal{G}_n$ is seen to be a λ-system which contains the class of intervals

$$\mathcal{I} := \{(0,t] : \ 0 < t \leq t\} \cup \{(-t, 0] : \ 0 < t \leq n\}$$

which forms a π-system. Hence by Dynkin's π-λ theorem (see e.g. Durrett [2]), $\mathcal{G}_n$ contains all Borel subsets of $[-n, n]$. Since $n \geq 1$ is arbitrary, and since we can write $N^*(B) = \lim_{n\to\infty} N^*(B \cap [-n, n])$, the mapping (5) is measurable for all Borel subsets of $\mathbf{R}$.

Remark 1. By an argument similar to (iii), it is easy to show that $\mathcal{M}_p$ is generated by mappings $M_p \ni N \mapsto X(t)$ for all t, where $X(t)$ is defined in (1).

Definition 1 *A* point process N_ω *is a random variable defined on a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *and taking values in the measurable space* $(M_p, \mathcal{M}_p)$.

Definition 2 *A point process N_ω is said to be* crudely stationary *if for any bounded interval I and for any $x \in \mathbf{R}$, $N_\omega(I)$ and $N_\omega(I + x)$ are identically distributed. Its* mean density *is the expectation value $m := \mathbf{E}[N_\omega((0,1])] \leq \infty$.*

Definition 3 *A point process N_ω is said to be* stationary *if for any $C \in \mathcal{M}_p$ and $x \in \mathbf{R}$, one has the identity*

$$\mathbf{P}(N_\omega(\cdot) \in C) = \mathbf{P}(N_\omega(x + \cdot) \in C) \ .$$

Obviously, N_ω is crudely stationary if it is stationary.

Remark 2. By another application of π-λ theorem, one can show without difficulty that N_ω is stationary if and only if for any finite family of Borel subsets $B_1, \ldots, B_n$ of $\mathbf{R}$, and of non-negative integers $k_1, \ldots, k_n$, the identity

$$\mathbf{P}(N_\omega(B_i) = k_i, \ i = 1, \ldots, n) = \mathbf{P}(N_\omega(x + B_i) = k_i, \ i = 1, \ldots, n)$$

holds for any $x \in \mathbf{R}$.

2 Basic theorems and their proofs.

Our argument is based on the following lemma, which is an immediate consequence of Definition 2.

Lemma 2 *Let the point process N_ω be crudely stationary. Then for any bounded interval I and for any non-negative integer k,*

$$\mathbf{P}(N_\omega(I) = k) = \int_0^1 \mathbf{P}(N_\omega(x+I) = k)dx = \mathbf{E}\Big[\int_0^1 \mathbf{1}_{\{N_\omega(x+I)=k\}}dx\Big] \ .$$

Proposition 1 (Khintchine's theorem) *For any crudely stationary point process N_ω, the limit*

$$\lambda := \lim_{h\searrow 0} \frac{1}{h}\mathbf{P}(N_\omega((0,h]) > 0)$$

exists and satisfies $\lambda \le m$. λ is called the intensity *of the point process N_ω.*

Proof. Let N_ω be represented as (2) and define the point process N^*_ω by (3). If we set $\nu(\omega) := N^*_\omega(0,1]$, it satisfies $x_{\nu(\omega)}(\omega) \le 1 < x_{\nu(\omega)+1}(\omega)$. Obviously we have

$$\{x \in (0,1] : \ N_\omega((x, x+h]) > 0\} = (0,1] \cap \Big[\bigcup_{j=1}^{\infty}[x_j(\omega) - h, x_j(\omega))\Big]$$

$$= \ (0,1] \cap \Big[\bigcup_{j=1}^{\nu(\omega)+1} J^\omega_j(h)\Big] = \sum_{j=1}^{\nu(\omega)+1}\Big[(0,1] \cap (J^\omega_j(h) \setminus J^\omega_{j-1}(h))\Big] \ ,$$

where we have set $J^\omega_j(h) := [x_j(\omega) - h, x_j(\omega))$ and $J_0 = \emptyset$. Hence

$$\begin{aligned} \frac{1}{h}\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h])>0\}}dx &= \frac{1}{h}\sum_{j=1}^{\nu(\omega)+1} |(0,1] \cap (J^\omega_j(h) \setminus J^\omega_{j-1}(h))| \\ &= \sum_{j=1}^{\nu(\omega)+1} \frac{1}{h}\{(1 \wedge x_j(\omega)) - (0 \vee x_{j-1}(\omega) \vee (x_j(\omega) - h)\}_+ \ , \end{aligned}$$

where for a Borel subset B of $\mathbf{R}$, $|B|$ denotes its Lebesgue measure and for a real number a, $a_+ := a \vee 0 = \max\{a, 0\}$ denotes its positive part. Now it is easy to see that for $1 \le j \le \nu(\omega)$,

$$\frac{1}{h}\{(1 \wedge x_j(\omega)) - (0 \vee x_{j-1}(\omega) \vee (x_j(\omega) - h)\}_+ \nearrow 1$$

as $h \searrow 0$, and that for $j = \nu(\omega) + 1$,

$$\frac{1}{h}\{(1 \wedge x_{\nu(\omega)+1}(\omega)) - (0 \vee x_{\nu(\omega)}(\omega) \vee (x_{\nu(\omega)+1}(\omega) - h))\}_+$$

is bounded by 1 and tends to 0 as $h \searrow 0$. Thus we can apply the monotone convergence theorem, the dominated convergence theorem and Lemma 2, to obtain

$$\begin{aligned} \frac{1}{h}\mathbf{P}(N_\omega((0,h]) > 0) &= \mathbf{E}\Big[\frac{1}{h}\sum_{j=1}^{\nu(\omega)+1} |(0,1] \cap (J_j^\omega \setminus J_{j-1}^\omega(h))|\Big] \\ &\to \mathbf{E}\Big[\sum_{j=1}^{\nu(\omega)} 1\Big] = \mathbf{E}\Big[N_\omega^*((0,1])\Big] , \end{aligned}$$

as $h \searrow 0$. Thus the desired limit λ exists and is equal to $\mathbf{E}[N_\omega^*((0,1])]$. Clearly it satisfies the inequality $\lambda \le \mathbf{E}[N_\omega((0,1])] = m$.

Corollary 1 *If N_ω is simple, then $\lambda = m$. When $m < \infty$, the converse is also true.*

Proof. N_ω is simple if and only if $N_\omega^* = N_\omega$ almost surely, which obviously implies $\lambda = m$. On the other hand, if $\lambda = m < \infty$, then

$$\mathbf{E}[N_\omega((0,1]) - N_\omega^*((0,1])] = m - \lambda = 0 .$$

But $N_\omega((0,1]) - N_\omega^*((0,1]) \ge 0$ in general, so that $N_\omega((0,1]) = N_\omega^*((0,1])$ almost surely. The same argument is valid if the interval $(0,1]$ is replaced by $(n, n+1]$, so that $N_\omega((n,n+1]) = N_\omega^*((n,n+1])$ almost surely for all integers n, and the simplicity of N_ω follows.

Remark 3. In the treatise by Daley and Vere-Jones [3], for example, Proposition 1 is proved in the following way: If we define $\phi(h) := \mathbf{P}(N_\omega((0,h]) > 0)$, then by the crude stationarity, we have for any positive h_1 and h_2,

$$\begin{aligned} \phi(h_1+h_2) &= \mathbf{P}(N_\omega((0,h_1+h_2]) > 0) = \mathbf{P}(N_\omega((0,h_1]) + N_\omega((h_1,h_1+h_2]) > 0) \\ &\le \mathbf{P}(N_\omega((0,h_1]) > 0) + \mathbf{P}(N_\omega((h_1,h_1+h_2]) > 0) = \phi(h_1) + \phi(h_2) , \end{aligned}$$

so that $\phi(h)$ is a sub-additive function defined on $[0,\infty)$ satisfying $\phi(0) = 0$. To show the existence of the intensity λ, it suffices to apply the following well known lemma.

Lemma 3 *Let $g(x)$ be a sub-additive function defined on $[0,\infty)$ such that $g(0) = 0$. Then one has*

$$\lim_{x\searrow 0} \frac{g(x)}{x} = \sup_{x>0} \frac{g(x)}{x} \le \infty .$$

However, this argument does not provide the representation $\lambda = \mathbf{E}[N_\omega^*((0,1])]$, so that the proof of Corollary 1 requires some extra work. Our proof above is closer to that of Leadbetter [5]. See also Chung [1].

Definition 4 *A crudely stationary point process N_ω is said to be* orderly *when*

$$\mathbf{P}(N_\omega((0,h]) \ge 2) = o(h) \quad (h \searrow 0) .$$

Proposition 2 (Dobrushin's theorem) *If a crudely stationary point process N_ω is simple and if $\lambda < \infty$, then N_ω is orderly.*

Proof. By Lemma 2, we can write

$$\mathbf{P}(N_\omega((0,h]) \geq 2) = \mathbf{E}\Big[\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h])\geq 2\}} dx\Big] \ .$$

As can be seen from the proof of Proposition 1, we have

$$\begin{aligned}
& \frac{1}{h}\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h])\geq 2\}} dx \leq \frac{1}{h}\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h])>0\}} dx \\
= & \sum_{j=1}^{\nu(\omega)} \frac{1}{h}|(0,1] \cap (J_j^\omega(h) \setminus J_{j-1}^\omega(h))| + \frac{1}{h}|(0,1] \cap (J_{\nu(\omega)+1}(h) \setminus J_{\nu(\omega)}(h)| \\
\leq & \ N_\omega^*((0,1]) + 1 \ ,
\end{aligned}$$

and

$$\lim_{h\searrow 0} \frac{1}{h}\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h])\geq 2\}} dx = \sharp\{j: \ x_j(\omega) \in (0,1], \ m_j(\omega) \geq 2\} \ .$$

Since $\mathbf{E}[N_\omega^*((0,1])] = \lambda < \infty$, we can apply the dominated convergence theorem, to obtain

$$\lim_{h\searrow 0} \frac{1}{h}\mathbf{P}(N_\omega((0,h]) \geq 2) = \mathbf{E}[\sharp\{j: \ x_j(\omega) \in (0,1], \ m_j(\omega) \geq 2\}] \ ,$$

which is equal to 0 if N_ω is simple.

Remark 4. The condition $\lambda < \infty$ cannot be dropped. For a counter example, see Exercise 3.3.2 of [3].

Proposition 3 (Korolyuk's theorem) *A crudely stationary, orderly point process is simple.*

Proof. By Fatou's lemma and the orderliness of N_ω,

$$\begin{aligned}
\mathbf{E}[\sharp\{j: \ x_j(\omega) \in (0,1], \ m_j(\omega) \geq 2\}] &= \mathbf{E}\Big[\liminf_{h\searrow 0} \frac{1}{h}\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h])\geq 2\}} dx\Big] \\
&\leq \liminf_{h\searrow 0} \frac{1}{h}\mathbf{P}(N_\omega((0,h]) \geq 2) = 0 \ ,
\end{aligned}$$

so that with probability one, N_ω has no multiple points in $(0,1]$. By crude stationarity, the above argument is also valid if $(0,1]$ is replaced by $(n, n+1]$ for any integer n. Hence N_ω is simple.

Proposition 4 *For a crudely stationary point process N_ω with finite intensity λ, the limits*

$$\lambda_k := \lim_{h \searrow 0} \frac{1}{h} \mathbf{P}(1 \le N_\omega((0,h]) \le k)$$

exists for $k = 1, 2, \ldots,$ and satisfy $\lambda_k \nearrow \lambda$ as $k \to \infty$. Moreover for $k = 1, 2, \ldots,$

$$\pi_k := \frac{\lambda_k - \lambda_{k-1}}{\lambda} = \lim_{h \searrow 0} \mathbf{P}(N_\omega((0,h]) = k \mid N_\omega((0,h]) > 0) \ ,$$

where we set $\lambda_0 := 0$.

Proof. As before, one has

$$\frac{1}{h} \int_0^1 \mathbf{1}_{\{1 \le N_\omega((x,x+h]) \le k\}} dx \le 1 + N_\omega^*((0,1]) \ ,$$

and

$$\lim_{h \searrow 0} \frac{1}{h} \int_0^1 \mathbf{1}_{\{1 \le N_\omega((x,x+h]) \le k\}} dx = \sharp\{j : \ x_j(\omega) \in (0,1], \ m_j(\omega) \le k\} \ .$$

Since $\lambda = \mathbf{E}[N_\omega^*((0,1])] < \infty$, we can apply the dominated convergence theorem and Lemma 2, to obtain

$$\lambda_k = \lim_{h \searrow 0} \frac{1}{h} \mathbf{E}\Big[\int_0^1 \mathbf{1}_{\{1 \le N_\omega((x,x+h]) \le k\}} dx\Big] = \mathbf{E}[\sharp\{j : \ x_j(\omega) \in (0,1], \ m_j(\omega) \le k\}] \ .$$

This representation of λ_k immediately gives

$$\lim_{k \to \infty} \lambda_k = \mathbf{E}[\sharp\{j : \ x_j(\omega) \in (0,1]\}] = \mathbf{E}[N^*((0,1])] = \lambda \ ,$$

by the monotone convergence theorem. The last statement of the proposition is obvious.

Corollary 2 *For a crudely stationary point process with finite intensity, we have*

$$\lambda \sum_{k=1}^{\infty} k \pi_k = \mathbf{E}[N_\omega((0,1])] = m \ .$$

3 The Palm measure

Let us assume that the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, on which our point process N_ω is defined, is equipped with a measurable flow $\{\theta_t\}_{t \in \mathbf{R}}$. Here a measurable flow $\{\theta_t\}$ is, by definition, a family of bijections $\theta_t : \Omega \to \Omega$ such that

(a) θ_0 is the identity mapping, and for any $s, t \in \mathbf{R}$, $\theta_s \circ \theta_t = \theta_{s+t}$ holds;

(b) the mapping $(t, \omega) \mapsto \theta_t(\omega)$ from $\mathbf{R} \times \Omega$ into Ω is jointly measurable with respect to $\mathcal{B}(\mathbf{R}) \times \mathcal{F}$, where $\mathcal{B}(\mathbf{R})$ is the Borel σ-algebra on $\mathbf{R}$.

Let us further assume that the relation

$$\int_{\mathbf{R}} N_{\theta_t\omega}(dx)\varphi(x) = \int_{\mathbf{R}} N_\omega(dx)\varphi(x-t) \tag{6}$$

holds for any $t \in \mathbf{R}$ and any continuous function φ with compact support. If the probability measure $\mathbf{P}$ is $\{\theta_t\}$-invariant in the sense $\mathbf{P} \circ \theta_t^{-1} = \mathbf{P}$ for all $t \in \mathbf{R}$, then by (6), our point process N_ω is stationary.

Definition 5 *The Palm measure of a point process $N_\omega(dx)$ is a measure kernel $Q(x, d\omega)$ on $\mathbf{R} \times \Omega$ such that for any jointly measurable, non-negative function $f(x,\omega)$, the relation*

$$\int_\Omega \mathbf{P}(d\omega) \int_{\mathbf{R}} N_\omega(dx) f(x,\omega) = \int_{\mathbf{R}} \lambda(dx) \int_\Omega Q(x, d\omega) f(x,\omega) \tag{7}$$

holds, where $\lambda(dx)$ is the mean measure of N_ω which is defined by $\lambda(B) = \mathbf{E}[N_\omega(B)]$ for $B \in \mathcal{B}(\mathbf{R})$ and which we assume to be finite for bounded Borel sets B.

Now let $u(t)$ be a probability density function on $\mathbf{R}$. Define a new probability measure $\mathbf{P}_u$ by

$$\int_\Omega \mathbf{P}_u(d\omega) g(\omega) = \int_{\mathbf{R}} u(t)dt \Big(\int_\Omega \mathbf{P}(d\omega) g(\theta_t\omega) \Big) , \tag{8}$$

where $g(\omega)$ is an arbitrary non-negative measurable function on Ω. Then the following result holds.

Theorem 1 *For any probability density $u(t)$ on $\mathbf{R}$, the Palm measure $Q_u(x, d\omega)$ exists for the point process N_ω defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P}_u)$.*

Proof. Let $f(x,\omega) \geq 0$ be jointly measurable on $\mathbf{R} \times \Omega$. Then we can rewrite the left hand side of (7) in the following way:

$$\begin{aligned}
\int_\Omega \mathbf{P}_u(d\omega) \int_{\mathbf{R}} N_\omega(dx) f(x,\omega) &= \int_{\mathbf{R}} u(t)dt \int_\Omega \mathbf{P}(d\omega) \int_{\mathbf{R}} N_{\theta_t\omega}(dx) f(x, \theta_t\omega) \\
&= \int_{\mathbf{R}} u(t)dt \int_\Omega \mathbf{P}(d\omega) \int_{\mathbf{R}} N_\omega(dx) f(x-t, \theta_t\omega) \\
&= \int_\Omega \mathbf{P}(d\omega) \int_{\mathbf{R}} N_\omega(dx) \int_{\mathbf{R}} u(t)dt f(x-t, \theta_t\omega) \\
&= \int_\Omega \mathbf{P}(d\omega) \int_{\mathbf{R}} N_\omega(dx) \int_{\mathbf{R}} u(x-s)ds f(s, \theta_{x-s}\omega) \\
&= \int_{\mathbf{R}} ds \int_\Omega \mathbf{P}(d\omega) \int_{\mathbf{R}} N_\omega(dx) u(x-s) f(s, \theta_{x-s}\omega) .
\end{aligned} \tag{9}$$

At this stage, take $f(x,\omega) = \varphi(x)$. Then (9) reduces to

$$\int_{\mathbf{R}} \varphi(s)\lambda(ds) = \int_{\mathbf{R}} \varphi(s)\ell_u(s)ds \tag{10}$$

with

$$\ell_u(s) = \int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_\omega(dx) u(x-s) \ . \tag{11}$$

If we define, for each $s \in \mathbf{R}$, the measure $Q_u(s, d\omega)$ on $(\Omega, \mathcal{F})$ by

$$\int_\Omega Q_u(s, d\omega) g(\omega) = \frac{\mathbf{1}_{(0,\infty)}(\ell_u(s))}{\ell_u(s)} \int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_\omega(dx) u(x-s) g(\theta_{x-s}\omega) \ , \tag{12}$$

then (9) takes the form of (7), and the theorem is proved.

When $\mathbf{P}$ is $\{\theta_t\}$-invariant, then we have $\mathbf{P}_u = \mathbf{P}$ for any probability density u on $\mathbf{R}$, and

$$\ell_u(s) = \int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_{\theta_s\omega}(dx) u(x) = \int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_\omega(dx) u(x) =: \ell > 0$$

is a constant. Moreover one can compute as

$$\begin{aligned} &\int_\Omega Q_u(s, d\omega) g(\omega) = \frac{1}{\ell} \int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_{\theta_s\omega}(dx) u(x) g(\theta_x\omega) \\ = \ &\frac{1}{\ell} \int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_{\theta_s\omega}(dx) u(x) g(\theta_{x-s}(\theta_s\omega)) = \frac{1}{\ell} \int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_\omega(dx) u(x) g(\theta_{x-s}\omega) \ . \end{aligned}$$

Hence if we define a measure $\hat{\mathbf{P}}(d\omega)$ on $(\Omega, \mathcal{F})$ by

$$\int_\Omega \hat{\mathbf{P}}(d\omega) g(\omega) = \int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_\omega(dx) u(x) g(\theta_x\omega) \ ,$$

then we get

$$Q_u(s, d\omega) = \frac{1}{\ell} (\hat{\mathbf{P}} \circ \theta_s)(d\omega) \ ,$$

and (7) can be written in the form

$$\int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_\omega(dx) f(x, \omega) = \int_\mathbf{R} dx \int_\Omega \hat{\mathbf{P}}(d\omega) f(x, \theta_{-x}\omega) \ , \tag{13}$$

which is the defining relation of the Palm measure in the stationary case (see [6]). (13) shows in particular that the definition of $\hat{\mathbf{P}}$ is independent of the choice of u.

Our consideration of the probability measure $\mathbf{P}_u$ is motivated by the following observation.

Proposition 5 *The probability measure* $\mathbf{P}$ *is* $\{\theta_t\}$*-invariant if and only if the following two conditions hold:*

(i) $\mathbf{P}_u = \mathbf{P}$ *for any probability density function* $u(t)$ *on* $\mathbf{R}$*;*

(ii) *the set* H *of all bounded measurable functions* $\varphi(\omega)$ *on* Ω *such that* $t \mapsto \varphi(\theta_t\omega)$ *is continuous for all* $\omega \in \Omega$ *is dense in* $L^2(\Omega, \mathbf{P})$.

Proof. The necessity of (i) is obvious. That (ii) also follows from the $\{\theta_t\}$-invariance of $\mathbf{P}$ is proved in [6] (see Lemma II. 3). To prove the sufficiency of (i) and (ii), fix an arbitrary $t_0 \in \mathbf{R}$ and take a sequence of probability density $\{u_n\}_n$ so that $u_n(t)dt \to \delta_{t_0}(dt)$ weakly. Now for any $\varphi \in H$, $t \mapsto \varphi(\theta_t\omega)$ is continuous and bounded by $\|\varphi\|_\infty := \sup_\Omega |\varphi(\omega)|$. Hence we can apply the dominated convergence theorem, to get

$$\begin{aligned}\int_\Omega \mathbf{P}(d\omega)\varphi(\theta_{t_0}\omega) &= \int_\Omega \mathbf{P}(d\omega)\left(\lim_{n\to\infty}\int_{\mathbf{R}} \varphi(\theta_t\omega)u_n(t)dt\right) \\ &= \lim_{n\to\infty}\int_{\mathbf{R}}\left(\int_\Omega \mathbf{P}(d\omega)\varphi(\theta_t\omega)\right)u_n(t)dt \\ &= \lim_{n\to\infty}\int_\Omega \mathbf{P}_{u_n}(d\omega)\varphi(\omega) = \int_\Omega \mathbf{P}(d\omega)\varphi(\omega)\end{aligned}$$

by condition (i). But if H is dense in $L^2(\Omega, \mathbf{P})$, we can approximate an arbitrary bounded measurable function $g(\omega)$ by the elements of H, to obtain

$$\int_\Omega \mathbf{P}(d\omega)g(\theta_{t_0}\omega) = \int_\Omega \mathbf{P}(d\omega)g(\omega)$$

for any $t_0 \in \mathbf{R}$. This sows the $\{\theta_t\}$-invariance of $\mathbf{P}$.

In most cases of application, Ω itself is a topological space with $\mathcal{F}$ the Baire σ-algebra generated by that topology and $t \mapsto \theta_t\omega$ is continuous for all $\omega \in \Omega$. In such a case, H contains the class $C_b(\Omega)$ of all bounded continuous functions on Ω, which is dense in $L^2(\Omega, \mathbf{P})$. Hence condition (ii) is not as restrictive as it may appear.

See [4] for a general treatment of stationary random measures on a topological group.

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