A note on Oseledec's theorem
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Consider a one-dimensional random Anderson model:

$$(h_\omega\psi)(n) = \psi(n-1) + \psi(n+1) + V_\omega(n)\psi(n)$$

and the transfer matrix associated to the equation $h_\omega\psi = E\psi$:

$$U_n(\omega) = S_n(\omega)S_{n-1}(\omega)\cdots S_1(\omega) \ ,$$

$$S_n(\omega) = \begin{bmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{bmatrix} .$$

Henceforth, we shall fix an energy value E and omit it from our notation. Here $\{V_\omega(n)\}_n$ is a sequence of independent, identically distributed (i.i.d.) random variables (or more generally an ergodic sequence of r.v.'s). Let us suppose for simpliciyt that $\{V_\omega(n)\}_n$ is bounded, namely that there is a constant W such that $|V_\omega(n)| \le W$ for all ω and n.

Since $\det U_n(\omega) = 1$, we have $\|U_n(\omega)\| \ge 1$. Hence by sub-additive ergodic theorem, there is a non-random constant $\gamma \ge 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|U_n(\omega)\| = \gamma \tag{1}$$

holds with probability one. We assume $\gamma > 0$, which is actually the case of i.i.d. random potential with a non-degenerate distribution.

From now on, we shall fix an ω for which (1) is true, and omit it from our notation. Let us decompose $U_n = U_n(\omega)$ in the form

$$U_n = L_n D_n K_n \ ,$$

where L_n and K_n are 2×2 real orthogonal matrices and D_n is a diagonal matrix with positive diagonal entries d_n and d_n^{-1} satisfying

$$\lim_{n\to\infty} \frac{1}{n} \log d_n = \gamma \ .$$

Let us now prove that the sequence of matrices

$$[U_n^T U_n]^{1/2n} = K_n^T D_n^{1/n} K_n$$

converges to a limit

$$\Lambda := K^T \begin{pmatrix} e^\gamma & 0 \\ 0 & e^{-\gamma} \end{pmatrix} K$$

with an orthogonal matrix K, and that if $\mathbf{v}$ is the eigenvector of Λ corresponding to the eigenvalue $e^{-\gamma}$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \|U_n \mathbf{v}\| = -\gamma \ .$$

Here for a matrix A, A^T denotes its transposition.

Lemma 1. Let $K_nK_{n+k}^T = (u_{ij}^{n,k})$. Then for any $M \geq 1$ and any $\delta \in (0, \gamma)$, one can choose an N so large that

$$\max\{|u_{12}^{n,k}|,\ |u_{21}^{n,k}|\} \leq e^{-(2\gamma-\delta)n}$$

holds for any $n \geq N$ and $k = 1, \cdots, M$.

Proof. From our assumption, there is a constant $A \geq 1$ such that $\|S_j\| \leq A$ for any $j = 1, 2, \cdots$. We can then choose an N such that as far as $n \geq N$, we have

$$A^M \leq e^{n\delta/4} \quad \text{and} \quad \left|\frac{1}{n}\log d_n - \gamma\right| \leq \frac{\delta}{4} \ .$$

The equality

$$U_{n+k} = S_{n+k} \cdots S_{n+1}U_n =: T_{n,k}U_n$$

can be written as

$$T_{n,k}L_nD_nK_n = L_{n+k}D_{n+k}K_{n+k} \ .$$

From this, we obtain

$$K_{n+k}K_n^T = D_{n+k}^{-1}(L_{n+k}^T T_{n,k}L_n)D_n \tag{2}$$

and

$$K_nK_{n+k}^T = D_n^{-1}(L_n^T T_{n,k}^{-1}L_{n+k})D_{n+k} \ . \tag{3}$$

If we let

$$C := L_{n+k}^T T_{n,k}L_n = (c_{ij}) \ , \quad \text{and} \quad \tilde{C} := L_n^T T_{n,k}^{-1}L_{n+k} = (\tilde{c}_{ij}) \ ,$$

then all c_{ij} and $\tilde{c}_{ij}$ are bounded in absolute value by $A^k \leq A^M \leq e^{n\delta/4}$, for $n \geq N$ and $k = 1, \cdots, M$.

Now from (3), we get

$$u_{12}^{n,k} = \tilde{C}_{12}d_n^{-1}d_{n+k}^{-1} \ ,$$

and hence for $n \geq N$ and $k = 1, \cdots, M$,

$$\begin{aligned} |u_{12}^{n,k}| &\leq e^{n\delta/4}e^{-n(\gamma-\delta/4)}e^{-(n+k)(\gamma-\delta/4)} \\ &= e^{-(2\gamma-\delta/2)n}e^{\frac{n}{4}\delta+\frac{k}{4}\delta-k\gamma} \leq e^{-(2\gamma-\delta)n} \ . \end{aligned}$$

Similarly we get

$$u_{21}^{n,k} = C_{12}d_n^{-1}d_{n+k}^{-1}$$

from (2), and $|u_{21}^{n,k}|$ can be estimated in the same way as above.

Lemma 2. There is a constant B such that for any $\delta \in (0, \gamma)$, one can choose N so large that for any $n \geq N$ and any $k \geq 1$, one has

$$\max\{|u_{12}^{n,k}|,\ |u_{21}^{n,k}|\} \leq Be^{-(2\gamma-\delta)n} \ .$$

Proof. In Lemma 1, let $M > (\log 2)/\gamma$ so that $2e^{-(2\gamma-\delta)}M < 1$ for any $\delta \in (0, \gamma)$. For a given $k \geq 1$, let $p \geq 0$ and $0 \leq q < M$ be defined by $k = pM + q$. Then since

$$K_nK_{n+pM+q}^T = (K_nK_{n+M}^T)(K_{n+M}K_{n+2M}^T) \cdots (K_{n+pM}K_{n+pM+q}^T) \ ,$$

we can write

$$u_{12}^{n,pM+q} = \sum_{j_1,\cdots,j_p=1}^{2} u_{1j_1}^{n,M} u_{j_1 j_2}^{n+M,M} \cdots u_{j_p,2}^{n+pM,q} \ .$$

Now if $(j_1, j_2, \cdots, j_p)$ is such that $j_k = 1$ and $j_{k+1} = \cdots = j_p = 2$ for some $0 \leq k \leq p$, then since we have $|u_{ij}^{n,k}| \leq 1$ in general and since we can choose an N according to Lemma 1 so that

$$|u_{12}^{n+kM,M}| \leq e^{-(2\gamma-\delta)(n+kM)} \ , \quad \text{and} \quad |u_{12}^{n+pM,q}| \leq e^{-(2\gamma-\delta)(n+pM)}$$

holds for $n \geq N$, we get the estimate

$$|u_{1j_1}^{n,M} u_{j_1 j_2}^{n+M,M} \cdots u_{j_p,2}^{n+pM,q}| \leq e^{-(2\gamma-\delta)(n+kM)} \ .$$

Noting that the number of the sequences $(j_1, \cdots, j_p)$ satisfying the condition stated above is 2^{k-1} for $1 \leq k \leq p$, we can estimate

$$\begin{aligned}
& |u_{12}^{n,pM+q}| \\
\leq \ & e^{-(2\gamma-\delta)n} + e^{-(2\gamma-\delta)(n+M)} + \cdots + 2^{p-2} e^{-(2\gamma-\delta)(n+(p-1)M)} + 2^{p-1} e^{-(2\gamma-\delta)(n+pM)} \\
= \ & e^{-(2\gamma-\delta)n} \Big\{ 1 + e^{-(2\gamma-\delta)M} + 2e^{-(2\gamma-\delta)2M} + \cdots + 2^{p-1} e^{-(2\gamma-\delta)pM} \Big\} \\
< \ & e^{-(2\gamma-\delta)n} \Big\{ 1 + e^{-(2\gamma-\delta)M} \sum_{j=0}^{\infty} \Big(2e^{-(2\gamma-\delta)M} \Big)^j \Big\} \\
< \ & e^{-(2\gamma-\delta)n} \Big\{ 1 + \frac{e^{-(2\gamma-\delta)M}}{1 - 2e^{-(2\gamma-\delta)M}} \Big\} < e^{-(2\gamma-\delta)n} \Big\{ 1 + \frac{1}{1 - 2e^{-\gamma M}} \Big\} \ .
\end{aligned}$$

Clearly $u_{21}^{n,pM+q}$ is bounded in the same way, and the assertion of the lemma holds with $B = 1 + 1/(1 - 2e^{-\gamma M})$.

Let us turn to the proof of the assertion stated earlier. The space $\mathcal{O}$ of orthogonal 2×2 matrices is closed and bounded (i.e. compact). Hence from the sequence $\{K_n\}$, we can extract a subsequence $\{K_{n_j}\}_j$ which converges to a limit $K \in \mathcal{O}$. Suppose $\{K_{n'_i}\}_i$ is another subsequence converging to a $\tilde{K} \in \mathcal{O}$. By Lemma 2, the $(1,2)$ and $(2,1)$ entry of $K_{n_j} K_{n'_i}^T$ is bounded by $Be^{-(2\gamma-\delta)n_j}$, as far as $N \leq n_j < n'_i$. If we let $n'_i \to \infty$ first, and then $n_j \to \infty$, we see that $K\tilde{K}^T$ is equal to the identity matrix. Hence $K = \tilde{K}$. From these considerations, we conclude that the sequence $\{K_n\}$ itself converges to a limit $K \in \mathcal{O}$, and that if we let $K_n K^T = (u_{ij}^n)$, then $|u_{ij}^n| \approx e^{-2n\gamma}$ for $(i,j) = (1,2),\ (2,1)$. Hence as $n \to \infty$, $U_n^T U_n = K_n^T D_n^{1/n} K_n$ converges to a matrix $\Lambda = K^T D K$, with a diagonal matrix D having e^{γ} and $e^{-\gamma}$ as diagonal entries. Let $\mathbf{v}$ be the eigenvector of Λ corresponding to the eigenvalue $e^{-\gamma}$. $\mathbf{v}$ is given by $\mathbf{v} = K^T \mathbf{e}_2$, with $\mathbf{e}_2 = [0,1]^T$. Now we can compute as follows:

$$\begin{aligned}
\|U_n \mathbf{v}\|^2 &= \langle U_n \mathbf{v}, U_n \mathbf{v} \rangle = \langle L_n D_n K_n K_n^T \mathbf{e}_2, L_n D_n K_n K_n^T \mathbf{e}_2 \rangle \\
&= \langle \mathbf{e}_2, K K_n^T D_n^2 K_n K^T \mathbf{e}_2 \rangle = d_n^2 (u_{12}^n)^2 + d_n^{-2} (u_{22}^n)^2 \ .
\end{aligned}$$

Since $d_n \approx e^{n\gamma}$, $u_{12}^n \approx e^{-2n\gamma}$, and $u_{22}^n \to 1$ as $n \to \infty$, this gives $\|U_n \mathbf{v}\|^2 \approx e^{-2n\gamma}$ as claimed.

Remark 1. Starting from a vector $\mathbf{v}$, the direction of $U_n\mathbf{v}$ should change frequently as n changes. It seems that the information on the direction of $U_n\mathbf{v}$ is contained in L_n, which did not played any role in the discussion above.

Remark 2. Recall that after the application of sub-additive ergodic theorem, we could fix an ω throughout. In other words, Oseledec's theorem is, in a sense, a deterministic (non-stochastic) result.

Remark 3. The above argument is due to

D. Ruelle: Ergodic theory of differentiable dynamical systems,
Publications mathémathiques de l'I.H.E.S., vol. 50 (1979), 27-58.

Ruelle's proof of Lemma 2 is incomprehensible to me. The present note is based on the following lecture note, with minor modifications:

F. Ledrappier: Quelques propriétés des exposants caractéritiques,
École d'Été de Probabilité de Saint-Flour XII-1982, Lecture Notes in Mathematics **1097**.